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Controllability and stabilization of finite dimensional systems

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Controllability of finite dimensional linear systems

Let $n, m \in \mathbb{N}^*$ and $T > 0$. Consider the following finite dimensional system:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \quad t \in (0, T), \\ x(0) = x^0. \end{cases} \tag{1}$$

In (1), A is a real $n \times n$ matrix, B is a real $n \times m$ matrix and x^0 a vector in $\mathbb{R}^n$. The function $x : [0, T] \longrightarrow \mathbb{R}^n$ represents the *state* and $u : [0, T] \longrightarrow \mathbb{R}^m$ the *control*. Both are vector functions of n and m components respectively depending exclusively on time t. Obviously, in practice $m \leq n$. The most desirable goal is, of course, controlling the system by means of a minimum number m of controls.

Given an initial datum $x^0 \in \mathbb{R}^n$ and a vector function $u \in L^2(0, T; \mathbb{R}^m)$, system (1) has a unique solution $x \in H^1(0, T; \mathbb{R}^n)$ characterized by the variation of constants formula:

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds, \quad \forall t \in [0, T]. \tag{2}$$

System (1) is **exactly controllable** in time $T > 0$ if given any initial and final one $x^0, x^1 \in \mathbb{R}^n$ there exists $u \in L^2(0, T, \mathbb{R}^m)$ such that the solution of (1) satisfies $x(T) = x^1$.

According to this definition the aim of the control process consists in driving the solution x of (1) from the initial state x^0 to the final one x^1 in time T by acting on the system through the control u.

Example 1. Consider the case

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3}$$

Then the system

$$x' = Ax + Bu$$

can be written as

$$\left\{ \begin{array}{c} x_1' = x_1 + u \\ x_2' = x_2, \end{array} \right.$$

or equivalently,

$$\left\{ \begin{array}{c} x_1' = x_1 + u \\ x_2 = x_2^0 e^t, \end{array} \right.$$

where $x^0 = (x_1^0, x_2^0)$ are the initial data.

This system is not controllable since the control u does not act on the second component x_2 of the state which is completely determined by the initial data x_2^0.,

Example 2. By the contrary, the equation of the harmonic oscillator is controllable

$$x'' + x = u. \tag{4}$$

The matrices A and B are now respectively

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Once again, we have at our disposal only one control u for both components x and y of the system. But, unlike in Example 1, now the control acts in the second equation where both components are present.

Define the set of reachable states

$$R(T, x^0) = \{x(T) \in \mathbb{R}^n : x \text{ solution of (1) with } u \in (L^2(0,T))^m\}, \tag{5}$$

the exact controllability property is equivalent to the fact that $R(T, x^0) = \mathbb{R}^n$ for any $x^0 \in \mathbb{R}^n$.

Observability property

Let A^* be the adjoint matrix of A, i.e. the matrix with the property that $\langle Ax, y\rangle = \langle x, A^* y\rangle$ for all $x, y \in \mathbb{R}^n$. Consider the following homogeneous adjoint system of (1):

$$\begin{cases} -\varphi' = A^*\varphi, \quad t \in (0,T) \\ \varphi(T) = \varphi_T. \end{cases} \tag{6}$$

This is an equivalent condition for exact controllability .

Lemma 1 *An initial datum $x^0 \in \mathbb{R}^n$ of (1) is driven to zero in time T by using a control $u \in L^2(0,T)$ if and only if*

$$\int_0^T \langle u, B^*\varphi\rangle dt + \langle x^0, \varphi(0)\rangle = 0, \quad \forall \varphi. \tag{7}$$

Proof: Let φ_T be arbitrary in $\mathbb{R}^n$ and φ the corresponding solution of (6). By multiplying (1) by φ and (6) by x we deduce that

$$\langle x', \varphi\rangle = \langle Ax, \varphi\rangle + \langle Bu, \varphi\rangle; \quad -\langle x, \varphi'\rangle = \langle A^*\varphi, x\rangle.$$

Hence,

$$\frac{d}{dt}\langle x, \varphi\rangle = \langle Bu, \varphi\rangle$$

which, after integration in time, gives that

$$\langle x(T), \varphi_T\rangle - \langle x^0, \varphi(0)\rangle = \int_0^T \langle Bu, \varphi\rangle dt = \int_0^T \langle u, B^*\varphi\rangle dt. \tag{8}$$

We obtain that $x(T) = 0$ if and only if (7) is verified for any $\varphi_T \in \mathbb{R}^n$.

Identity (7) is in fact an optimality condition for the critical points of the quadratic functional $J : \mathbb{R}^n \to \mathbb{R}^n$,

$$J(\varphi_T) = \frac{1}{2}\int_0^T |\ B^*\varphi\ |^2\, dt + \langle x^0, \varphi(0)\rangle$$

where φ is the solution of the adjoint system (6) with initial data φ_T at time $t = T$.

More precisely:

Lemma 2 *Suppose that J has a minimizer $\widehat{\varphi}_T \in \mathbb{R}^n$ and let $\widehat{\varphi}$ be the solution of the adjoint system (6) with initial data $\widehat{\varphi}_T$. Then*

$$u = B^*\widehat{\varphi} \tag{9}$$

is a control of system (1) with initial data x^0.

Proof: If $\widehat{\varphi}_T$ is a point where J achieves its minimum value, then

$$\lim_{h\to 0}\frac{J\left(\widehat{\varphi}_T + h\varphi_T\right) - J\left(\widehat{\varphi}_T\right)}{h} = 0, \quad \forall \varphi_T \in \mathbb{R}^n.$$

This is equivalent to

$$\int_0^T \langle B^*\widehat{\varphi}, B^*\varphi\rangle dt + \langle x^0, \varphi(0)\rangle = 0, \quad \forall \varphi_T \in \mathbb{R}^n,$$

which, in view of Lemma 1, implies that $u = B^*\widehat{\varphi}$ is a control for (1).

This is a variational method to obtain the control as a minimum of the functional J. The controls we obtian this way are smooth but this is not the unique possible functional allowing to build the control. By modifying it conveniently, other types of controls (for instance *bang-bang* ones) can be obtained.

System (6) is said to be **observable** in time $T > 0$ if there exists $c > 0$ such that

$$\int_0^T | B^*\varphi |^2 \, dt \geq c \, | \varphi(0) |^2, \qquad (10)$$

for all $\varphi_T \in \mathbb{R}^n$, φ being the corresponding solution of (6).

In the sequel (10) will be called the **observation** or **observability inequality**. It guarantees that the solution of the adjoint problem at $t = 0$ is uniquely determined by the observed quantity $B^*\varphi(t)$ for $0 < t < T$. In other words, the information contained in this term completely characterizes the solution of (6).

The following remark is very important in the context of finite dimensional control. Unfortunately (?) it is not true for infinite-dimensional systems (PDE, distributed parameter systems).

Inequality (10) is equivalent to the following unique continuation principle:

$$B^*\varphi(t) = 0, \ \forall t \in [0, T] \Rightarrow \varphi_T = 0. \tag{11}$$

This is an uniqueness or unique continuation property.

UNIQUE CONTINUATION $\rightarrow$

OBSERVABILITY INEQUALITY $\rightarrow$

CONTROLLABILITY,

WITH A CONSTRUCTIVE PROCEDURE TO BUILD CONTROLS BY MINIMIZING A COERCIVE FUNCTIONAL.

Kalman's controllability condition

What about the observability property. Are there algebraic conditions on the state matrix A and the control one B for it to be true?

The following classical result is due to R. E. Kalman and gives a complete answer to the problem of exact controllability of finite dimensional linear systems. It shows, in particular, that the time of control is irrelevant, something which is far from being true in the context of PDE.

Theorem 1 *System* (1) *is exactly controllable in some time* T *if and only if*

$$\text{rank}\,[B,\ AB, \cdots, A^{n-1}B] = n. \tag{12}$$

Consequently, if system (1) is controllable in some time $T > 0$ *it is controllable in any time.*

Remark 1 *From now on we shall simply say that* (A, B) *is controllable if (12) holds. The matrix* $[B, AB, \cdots, A^{n-1}B]$ *will be called the controllability matrix.*

Proof of Theorem 1: "$\Rightarrow$" Suppose that rank$([B,\ AB, \cdots, A^{n-1}B]) <$ n.

Then the rows of the controllability matrix $[B, AB, \cdots, A^{n-1}B]$ are linearly dependent and there exists a vector $v \in \mathbb{R}^n$, $v \neq 0$ such that

$$v^*[B,\ AB, \cdots, A^{n-1}B] = 0.$$

Then $v^*B = v^*AB = \cdots = v^*\,A^{n-1}B = 0$. From Cayley-Hamilton Theorem we deduce that there exist constants $c_1, \cdots, c_n$ such that, $A^n = c_1A^{n-1} + \cdots + c_nI$ and therefore $v^*A^nB = 0$, too. In fact, it follows that $v^*A^kB = 0$ for all $k \in \mathbb{N}$ and consequently $v^*e^{At}B = 0$ for all t as well. But, from the variation of constants formula, the solution x of (1) satisfies

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds. \tag{13}$$

Therefore

$$\langle v, x(T)\rangle = \langle v, e^{AT}x^0\rangle + \int_0^T \langle v, e^{A(T-s)}Bu(s)\rangle ds = \langle v, e^{AT}x^0\rangle.$$

Hence, $\langle v, x(t)\rangle$ is independent of t. This shows that the projection of the solution x on v is independent of the value of the control u. Hence, the system is not controllable.

Remark 2 *The conservation property for the quantity $\langle v, x\rangle$ we have just proved holds for any vector v for which $v[B, AB, \cdots, A^{n-1}B] = 0$. Thus, if the rank of the matrix $[B, AB, \cdots, A^{n-1}B]$ is $n-k$, the reachable set that $x(T)$ runs is an affine subspace of $\mathbb{R}^n$ of dimension $n-k$.*

"$\Leftarrow$" Suppose now that rank$([B, AB, \cdots, A^{n-1}B]) = n$. It is sufficient to show that system (6) is observable.

Assume $B^*\varphi = 0$ and $\varphi(t) = e^{A^*(T-t)}\varphi_T$, it follows that $B^* e^{A^*(T-t)}\varphi_T \equiv 0$ for all $0 \leq t \leq T$. By computing the derivatives of this function in $t = T$ we obtain that

$$B^*[A^*]^k \varphi_T = 0 \quad \forall k \geq 0.$$

But since $\text{rank}(\left[B,\, AB, \cdots, A^{n-1}B\right]) = n$ we deduce that

$$\text{rank}(\left[B^*,\, B^*A^*, \cdots, B^*(A^*)^{n-1}\right]) = n$$

and therefore $\varphi_T = 0$. Hence, (11) is verified and the proof of Theorem 1 is now complete.

Remark 3 *The set of controllable pairs (A, B) is open and dense. This means that*

- *Most systems are controllable;*

- *The controllability property is robust, i. e. it is invariant under small perturbations of A and/or B.*

When controllability holds the norm of the control is proportional to the distance between $e^{AT}x^0$ (the state freely attained by the system in the absence of control, i. e. with $u = 0$) and the objective x^1,

$$\| u \|_{L^2(0,T)} \leq C|e^{AT}x^0 - x^1| \tag{14}$$

for any initial data x^0 and final objective x^1.

Remark 4 *Linear scalar equations of any order provide examples of systems of arbitrarily large dimension that are controllable with only one control:* k

$$x^{(k)} + a_1 x^{(k-1)} + \ldots + a_{k-1} x = u.$$

Exercise: Check that the Kalman condition is fulfilled in this case.

Bang-bang controls

Let us consider the particular case

$$B \in \mathcal{M}_{n\times 1}, \tag{15}$$

i. e. $m = 1$, in which only one control $u : [0, T] \to \mathbb{R}$ is available and B is a column vector.

To build bang-bang controls it is convenient to consider the quadratic functional:

$$J_{bb}(\varphi^0) = \frac{1}{2}\left[\int_0^T \mid B^*\varphi \mid dt\right]^2 + \langle x^0, \varphi(0)\rangle \tag{16}$$

where φ is the solution of the adjoint system (6) with initial data φ_T.

The same argument as above show that J_{bb} is also continuous and coercive. It follows that J_{bb} attains a minimum in some point $\widehat{\varphi}_T \in \mathbb{R}^n$.

The optimality condition (the Euler-Lagrange equations) its minimizers satisfy:

$$\int_0^T \mid B^*\widehat{\varphi} \mid dt \int_0^T \mathrm{sgn}\,(B^*\widehat{\varphi}) B^*\psi(t)dt + \langle x^0, \varphi(0)\rangle = 0$$

for all $\varphi_T \in \mathbb{R}$, where φ is the solution of the adjoint system (6) with initial data φ_T.

The control we are looking for is

$$u = \int_0^T \mid B^* \widehat{\varphi} \mid dt \,\mathrm{sgn}\,(B^* \widehat{\varphi})$$

where $\widehat{\varphi}$ is the solution of (6) with initial data $\widehat{\varphi}_T$.

Note that the control u is of bang-bang form. Indeed, u takes only two values $\pm \int_0^T \mid B^* \widehat{\varphi} \mid dt$. The control switches from one value to the other finitely many times when the function $B^* \widehat{\varphi}$ changes sign.

The control $u_\infty = \int_0^T \mid B^* \widehat{\varphi} \mid dt \,\mathrm{sgn}\,(B^* \widehat{\varphi})$ obtained by minimizing the functional J_{bb} has minimal $L^\infty(0,T)$ norm among all possible controls. and the proof finishes.

Stabilization of finite dimensional linear systems

The controls we have obtained so far are the so called open loop controls. In practice, it is interesting to get closed loop or feedback controls, so that its value is realted in real time with the state itself.la

In this section we assume that A is a skew-adjoint matrix, i. e. $A^* = -A$. In this case, $< Ax, x >= 0$. Consider the system

$$\begin{cases} x' = Ax + Bu \\ x(0) = x^0. \end{cases} \tag{17}$$

When $u \equiv 0$, the energy of the solution of (17) is conserved. Indeed, by multiplying (17) by x, if $u \equiv 0$, one obtains

$$\frac{d}{dt}|x(t)|^2 = 0. \tag{18}$$

Hence,

$$|x(t)| = |x^0|, \quad \forall t \geq 0. \tag{19}$$

The problem of *stabilization* can be formulated in the following way. Suppose that the pair (A, B) is controllable. We then look for a matrix L such that the solution of system (17) with the *feedback* control law

$$u(t) = Lx(t) \tag{20}$$

has **a uniform exponential decay**, i.e. there exist $c > 0$ and $\omega > 0$ such that

$$|x(t)| \leq ce^{-\omega t}|x^0| \tag{21}$$

for any solution.

Note that, according to the law (20), the control u is obtained in real time from the state x.

In other words, we are looking for matrices L such that the solution of the system

$$x' = (A + BL)x = Dx \tag{22}$$

has an uniform exponential decay rate.

Remark that we cannot expect more than (21). Indeed, for instance, the solutions of (22) may not satisfy $x(T) = 0$ in finite time T. Indeed, if it were the case, from the uniqueness of solutions of (22) with final state 0 in $t = T$, it would follow that $x^0 \equiv 0$.

Theorem 2 *If A is skew-adjoint and the pair (A, B) is controllable then $L = -B^*$ stabilizes the system, i.e. the solution of*

$$\left\{ \begin{array}{c} x' = Ax - BB^*x \\ x(0) = x^0 \end{array} \right. \tag{23}$$

has an uniform exponential decay (21).

Proof: With $L = -B^*$ we obtain that

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 = - < BB^*x(t), x(t) >= - \mid B^*x(t) \mid^2 \leq 0.$$

Hence, the norm of the solution decreases in time.

Moreover,

$$|x(T)|^2 - |x(0)|^2 = -2 \int_0^T \mid B^*x \mid^2 dt. \tag{24}$$

To prove the uniform exponential decay it is sufficient to show that there exist $T > 0$ and $c > 0$ such that

$$|x(0)|^2 \leq c \int_0^T \mid B^* x \mid^2 dt \tag{25}$$

for any solution x of (23). Indeed, from (24) and (25) we would obtain that

$$|x(T)|^2 - |x(0)|^2 \leq -\frac{2}{c}|x(0)|^2 \tag{26}$$

and consequently

$$|x(T)|^2 \leq \gamma |x(0)|^2 \tag{27}$$

with

$$\gamma = 1 - \frac{2}{c} < 1. \tag{28}$$

Hence,

$$|x(kT)|^2 \le \gamma^k |x^0|^2 = e^{(\ln \gamma)k} |x^0|^2 \quad \forall k \in \mathbb{N}. \tag{29}$$

Now, given any $t > 0$ we write it in the form $t = kT + \delta$, with $\delta \in [0, T)$ and $k \in \mathbb{N}$ and we obtain that

$$|x(t)|^2 \le |x(kT)|^2 \le e^{-|\ln(\gamma)|k} |x^0|^2 =$$

$$= e^{-|\ln(\gamma)|\left(\frac{t}{T}\right)} e^{|\ln(\gamma)|\frac{\delta}{T}} |x^0|^2 \le \frac{1}{\gamma} e^{-\frac{|\ln(\gamma)|}{T} t} |x^0|^2.$$

We have obtained the desired decay result (21) with

$$c = \frac{1}{\gamma}, \, \omega = \frac{|\ln(\gamma)|}{T}. \tag{30}$$

To prove (25) we decompose the solution x of (23) as $x = \varphi + y$ with φ and y solutions of the following systems:

$$\begin{cases} \varphi' = A\varphi \\ \varphi(0) = x^0, \end{cases} \tag{31}$$

and

$$\begin{cases} y' = Ay - BB^*x \\ y(0) = 0. \end{cases} \tag{32}$$

Remark that, since A is skew-adjoint, (31) is exactly the adjoint system (6) except for the fact that the initial data are taken at $t = 0$.

As we have seen in the proof of Theorem 1, the pair (A, B) being controllable, the following observability inequality holds for system

(31):

$$|x^0|^2 \leq C \int_0^T \mid B^*\varphi \mid^2 dt. \tag{33}$$

Since $\varphi = x - y$ we deduce that

$$|x^0|^2 \leq 2C \left[\int_0^T \mid B^* x \mid^2 dt + \int_0^T \mid B^* y \mid^2 dt \right].$$

On the other hand, it is easy to show that the solution y of (32) satisfies:

$$\frac{1}{2}\frac{d}{dt} \mid y \mid^2 = -\langle B^* x,\, B^* y \rangle \leq |B^* x|\, |B^*|\, |y| \leq \frac{1}{2}\left(|y|^2 + |B^*|^2 |B^* x|^2 \right).$$

From Gronwall's inequality we deduce that

$$| \, y(t) \, |^2 \leq |B^*|^2 \int_0^t e^{t-s} \, | \, B^* x \, |^2 \, ds \leq |B^*|^2 e^T \int_0^T | \, B^* x \, |^2 \, dt \tag{34}$$

and consequently

$$\int_0^T | \, B^* y \, |^2 \, dt \leq |B|^2 \int_0^T | \, y \, |^2 \, dt \leq T |B|^4 e^T \int_0^T | \, B^* x \, |^2 \, dt.$$

Finally, we obtain that

$$| \, x^0 \, |^2 \leq 2C \int_0^T | \, B^* x \, |^2 \, dt + C |B^*|^4 e^T T \int_0^T | \, B^* x \, |^2 \, dt \leq C' \int_0^T | \, B^* x \, |^2 \, dt$$

and the proof of Theorem 2 is complete.

Example: Consider the damped harmonic oscillator:

$$mx'' + Rx + kx' = 0, \tag{35}$$

where m, k and R are positive constants.

Note that (35) may be written in the equivalent form

$$mx'' + Rx = -kx'$$

which indicates that an applied force, proportional to the velocity of the point-mass and of opposite sign, is acting on the oscillator.

It is easy to see that the solutions of this equation have an exponential decay property. Indeed, it is sufficient to remark that the two characteristic roots have negative real part. Indeed,

$$mr^2 + R + kr = 0 \Leftrightarrow r_\pm = \frac{-k \pm \sqrt{k^2 - 4mR}}{2m}$$

and therefore

$$\operatorname{Re} r_{\pm} = \begin{cases} -\frac{k}{2m} & \text{if } k^2 \le 4mR \\ -\frac{k}{2m} \pm \sqrt{\frac{k^2}{4m} - \frac{R}{2m}} & \text{if } k^2 \ge 4mR. \end{cases}$$

We observe here the classical overdamping phenomenon. Contradicting a first intuition it is not true that the decay rate increases when the value of the damping parameter k increases.

If (A, B) is controllable, we have proved the uniform stability property of the system (17), under the hypothesis that A is skew-adjoint. However, this property holds even if A is an arbitrary matrix. More precisely, we have

Theorem 3 *If (A, B) is controllable then it is also stabilizable. Moreover, it is possible to prescribe any complex numbers λ_1, λ_2,...,λ_n as the eigenvalues of the closed loop matrix $A + BL$ by an appropriate choice of the feedback matrix L so that the decay rate may be made arbitrarily fast.*

This result is not in contradiction with the behavior we observed above on the harmonic oscillator (the overdamping phenomenon). In order to obntian the arbitrarily fast decay one needs to use all components of the state on the feedback law!

We have shown that

- Controllability and observabilitiy are equivalent notions (Wiener's cybernetics);
- Both hold for all T if and only if the Kalman rank condition is fulfilled.
- The controls may be obtained as min imizers of suitable quadratic functionals over the space of solutions of the adjoint system.
- There are very many controls: smooth ones, in bang-bang form,...

- Controllable systems are stabillizable by means of closed loop or feedback controls.

References: See in particular de the book by E. Trélat in french, which provides a very nice introduction to the control of finite-dimensional systems.