Mathematical Morphology

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A few references

- J. Serra, *Image Analysis and Mathematical Morphology*, Academic Press, New-York, 1982.
- J. Serra (Ed.), *Image Analysis and Mathematical Morphology, Part II: Theoretical Advances*, Academic Press, London, 1988.
- P. Soille, *Morphological Image Analysis*, Springer-Verlag, Berlin, 1999.

Shape or spatial relations?

Simplifying and selecting relevant information...

Introduction

- Origin: study of porous media
- Principle: study of objects (images) based on:
  - shape, geometry, topology
  - grey levels, colors
  - neighborhood information
- Mathematical bases:
  - set theory
  - topology
  - geometry
  - algebra (lattice theory)
  - probabilities, random closed sets
  - functions
- Main characteristics:
  - non linear
  - non invertible
  - strong properties
  - associated algorithms

Tools for

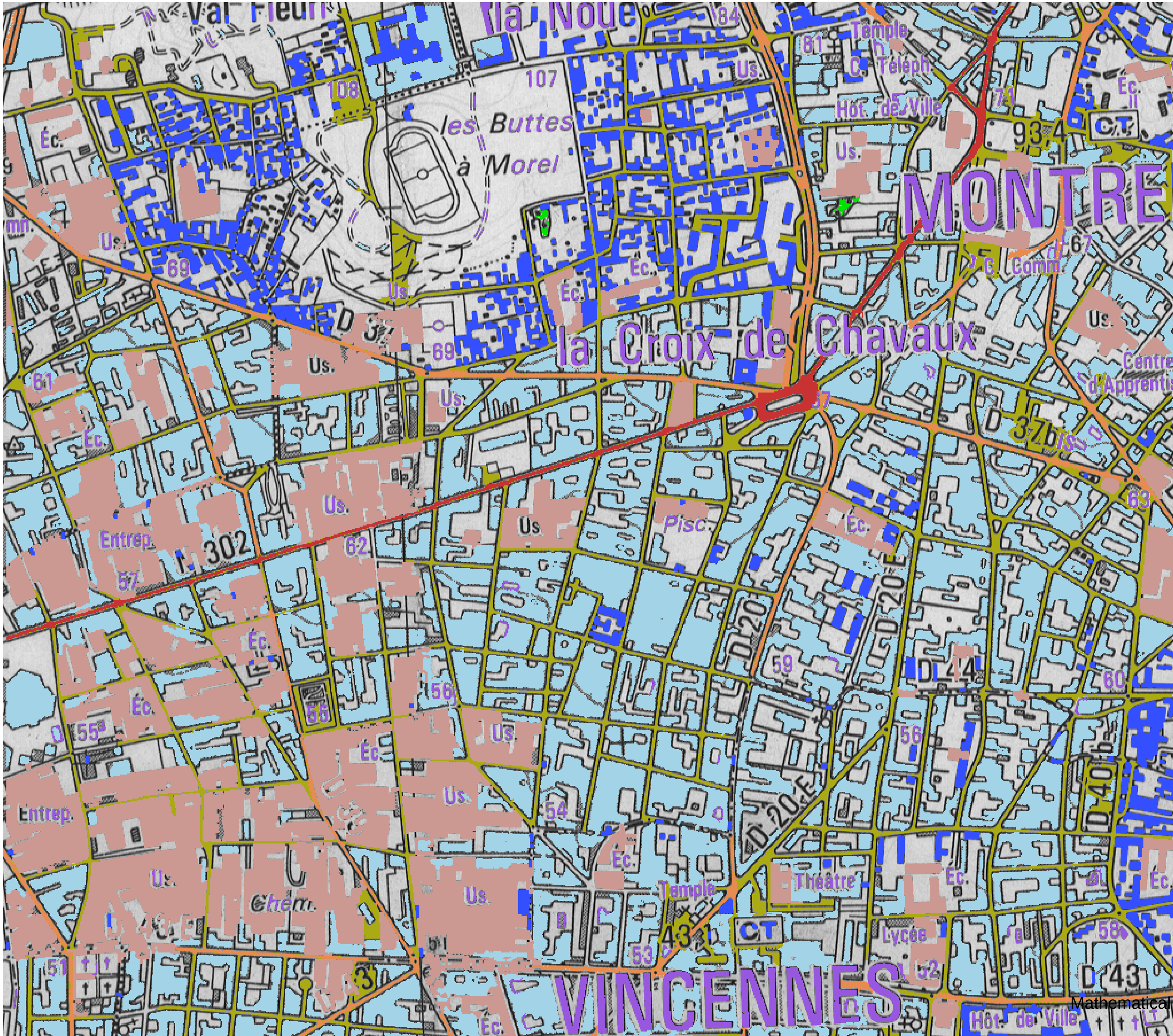
- filtering
- segmentation
- measures (distances, granulometry, integral geometry, topology, stochastic processes...)
- texture analysis
- shape recognition
- scene interpretation
- ...

Applications in numerous domains

Example

la Noue
Temple
Hôt. de Ville
les Buttes
à Morel
C. Comm.
la Croix de Chavaux
Centre
D 37bis
D 37
N 302
Entrep.
Pisc.
D 20
D 20 E
D 44
D 39
D 43 E
Chem.
Temple
Théâtre
Lycée
VINCENNES
D 43

Example



la Noue
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D 37bis

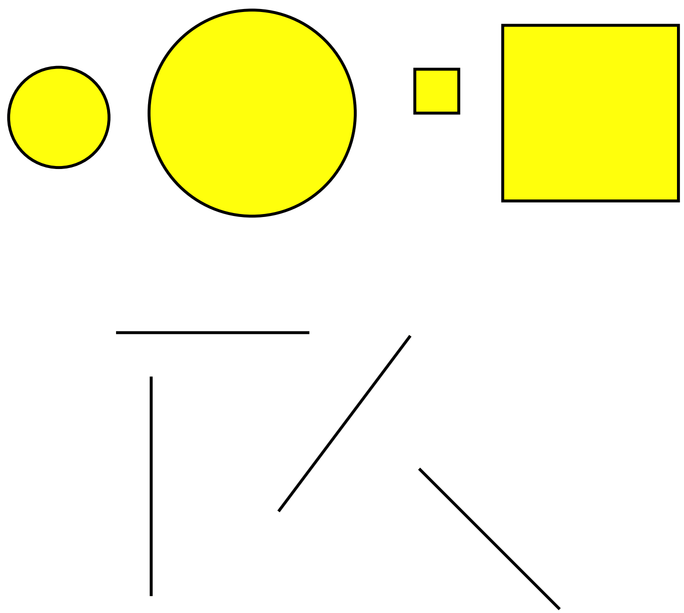
Four fundamental principles

1. Compatibility with translations
2. Compatibility with scaling
3. Local knowledge
4. Continuity (semi-continuity)

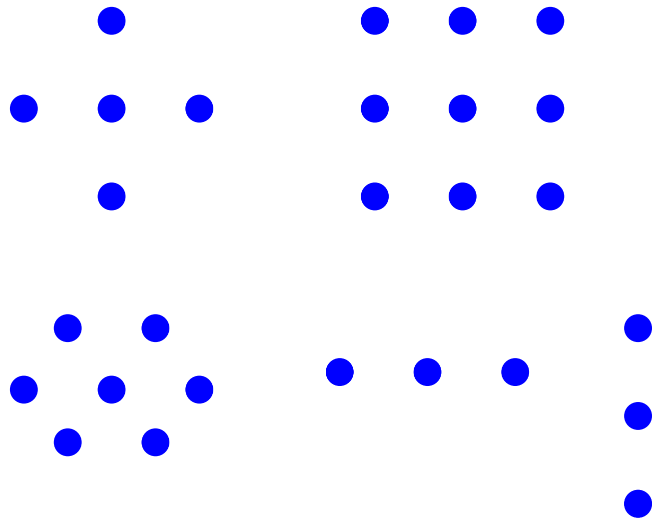
Structuring element

- shape
- size
- origin (not necessarily in B)
- examples:

Continuous:



Digital:



Binary dilation

- Minkowski addition:

$$X \oplus Y = \{x + y \ / \ x \in X, y \in Y\}$$

- Binary dilation:

$$\begin{aligned} D(X,B) &= X \oplus \check{B} = \{x + y \ / \ x \in X, y \in \check{B}\} \ (or \ = X \oplus B) \\ &= \bigcup_{x \in X} \check{B}_x = \{x \in \mathbb{R}^n \ / \ B_x \cap X \neq \emptyset\} \end{aligned}$$

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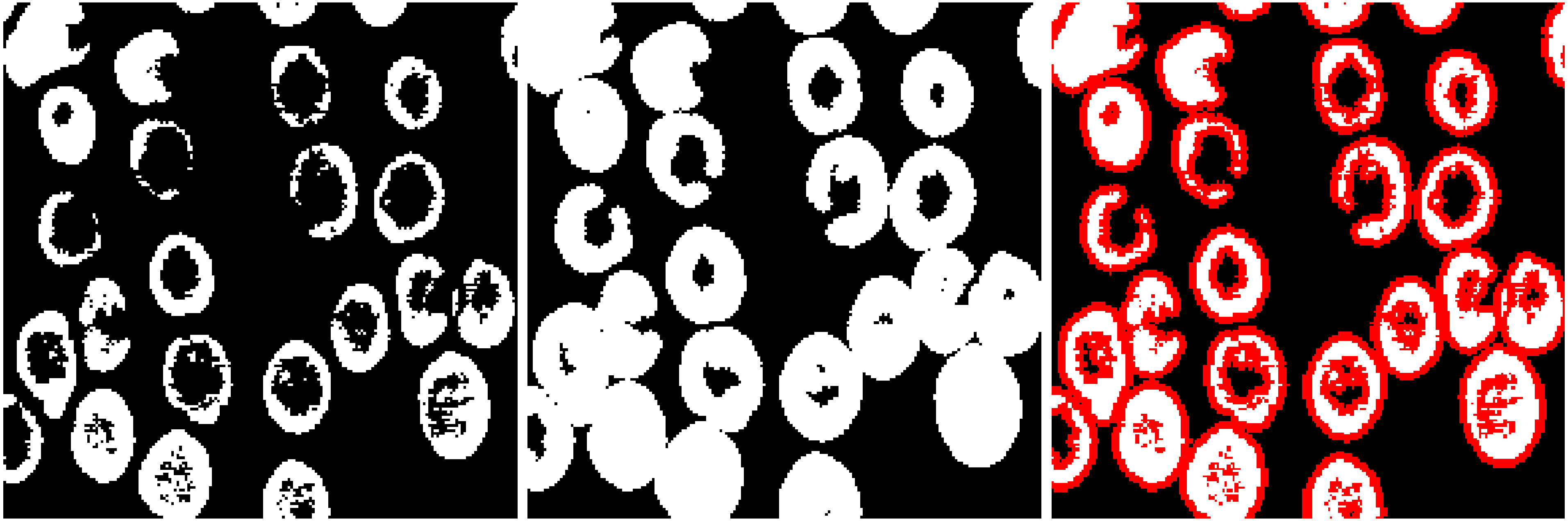
Properties of dilation:

- extensive ($X \subseteq D(X,B)$) if $O \in B$;
- increasing ($X \subseteq Y \Rightarrow D(X,B) \subseteq D(Y,B)$);
- $B \subseteq B' \Rightarrow D(X,B) \subseteq D(X,B')$;
- commutes with union, not with intersection:

$$D(X \cup Y, B) = D(X,B) \cup D(Y,B), \quad D(X \cap Y, B) \subseteq D(X,B) \cap D(Y,B);$$

- iterativity property: $D[D(X,B),B'] = D(X, B \oplus B')$.

Example of dilation



Binary erosion

$$
\begin{aligned}
E(X,B) &= \{x \in \mathbb{R}^n \;/\; B_x \subseteq X\} \\
&= \{x \;/\; \forall y \in B, x + y \in X\} = X \ominus \check{B}.
\end{aligned}
$$

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\end{aligned}
$$

Properties of erosion:

- duality of erosion and dilation with respect to complementation:

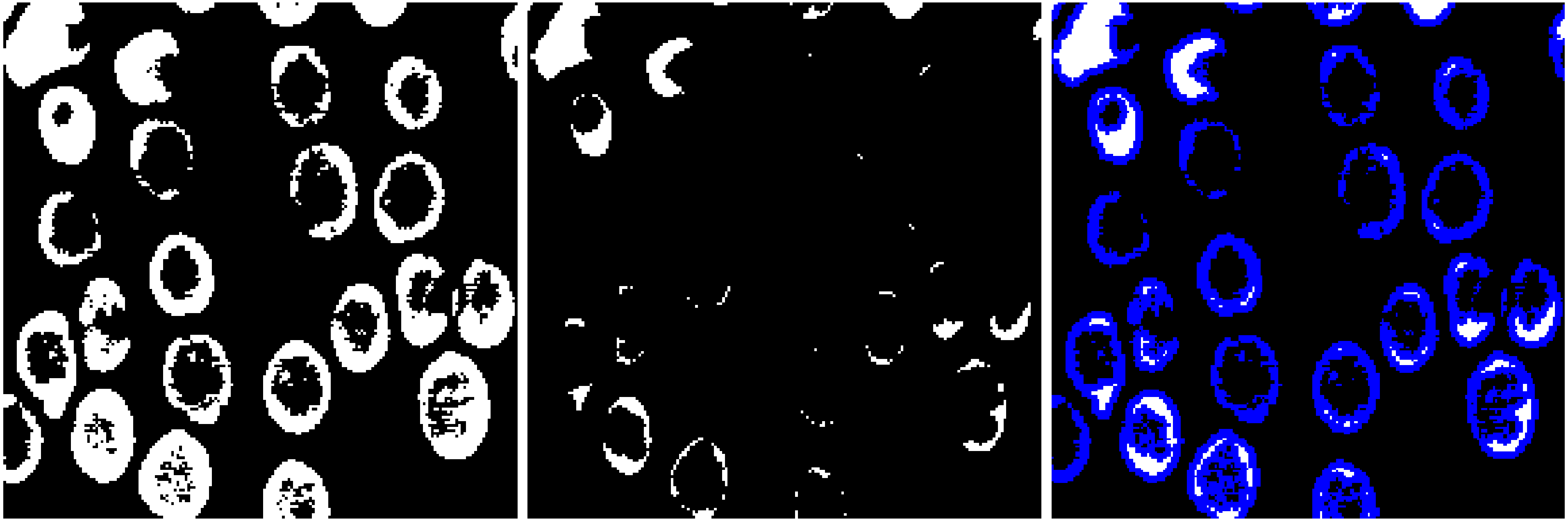
$$E(X,B) = [D(X^C,B)]^C \quad (or\ E(X,B) = [D(X^C,\check{B})]^C)$$

- anti-extensive ($E(X,B) \subseteq X$) if $O \in B$;
- increasing ($X \subseteq Y \Rightarrow E(X,B) \subseteq E(Y,B)$);
- $B \subseteq B' \Rightarrow E(X,B') \subseteq E(X,B)$;
- commutes with intersection, not with union:

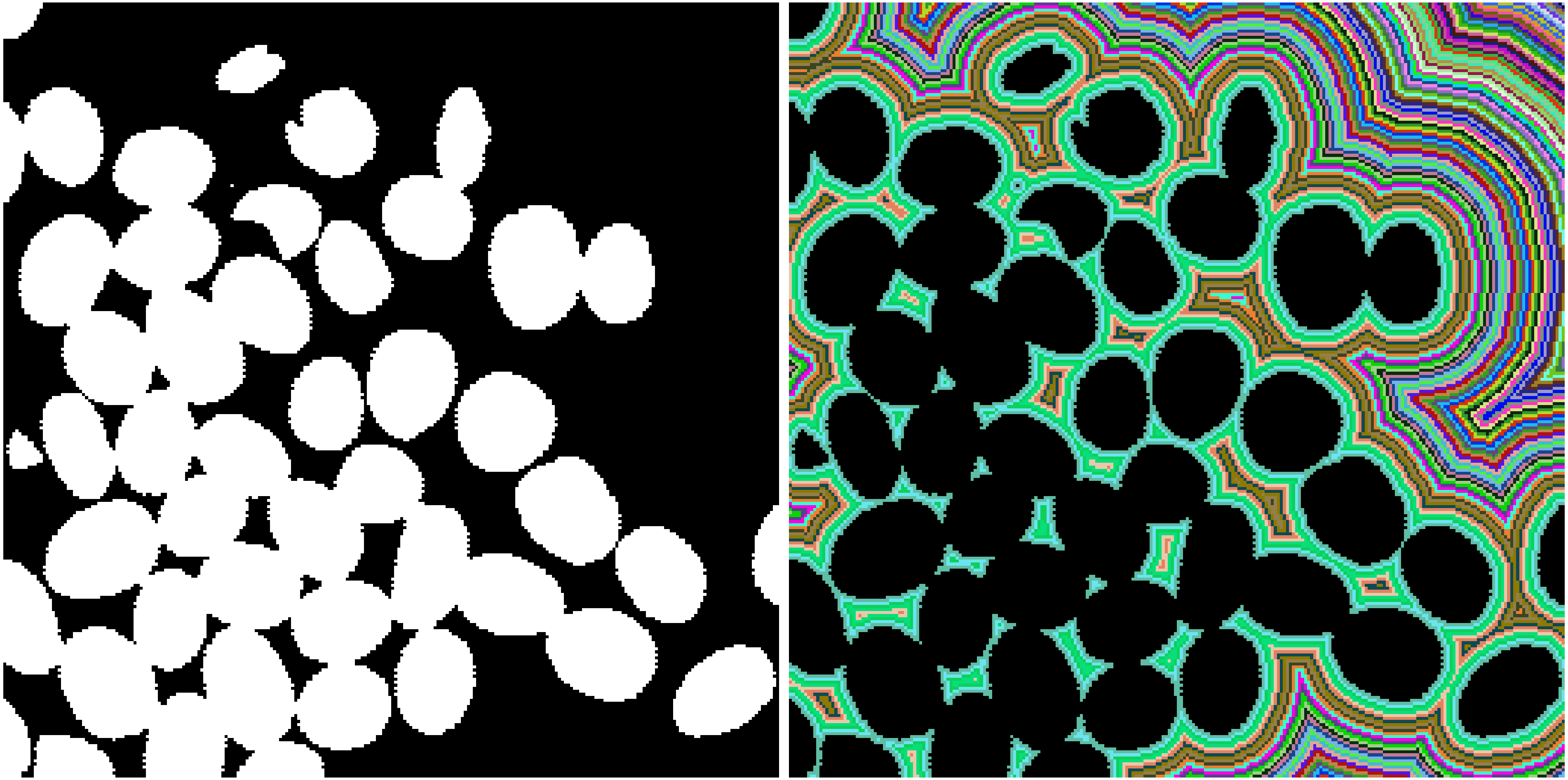
$$E[(X \cap Y),B] = E(X,B) \cap E(Y,B), \quad E[(X \cup Y),B] \supseteq E(X,B) \cup E(Y,B);$$

- iterativity property: $E[E(X,B),B'] = E(X,B \oplus B')$.
- $D[E(X,B),B'] \subseteq E[D(X,B'),B]$.

Example of erosion



Links with distances



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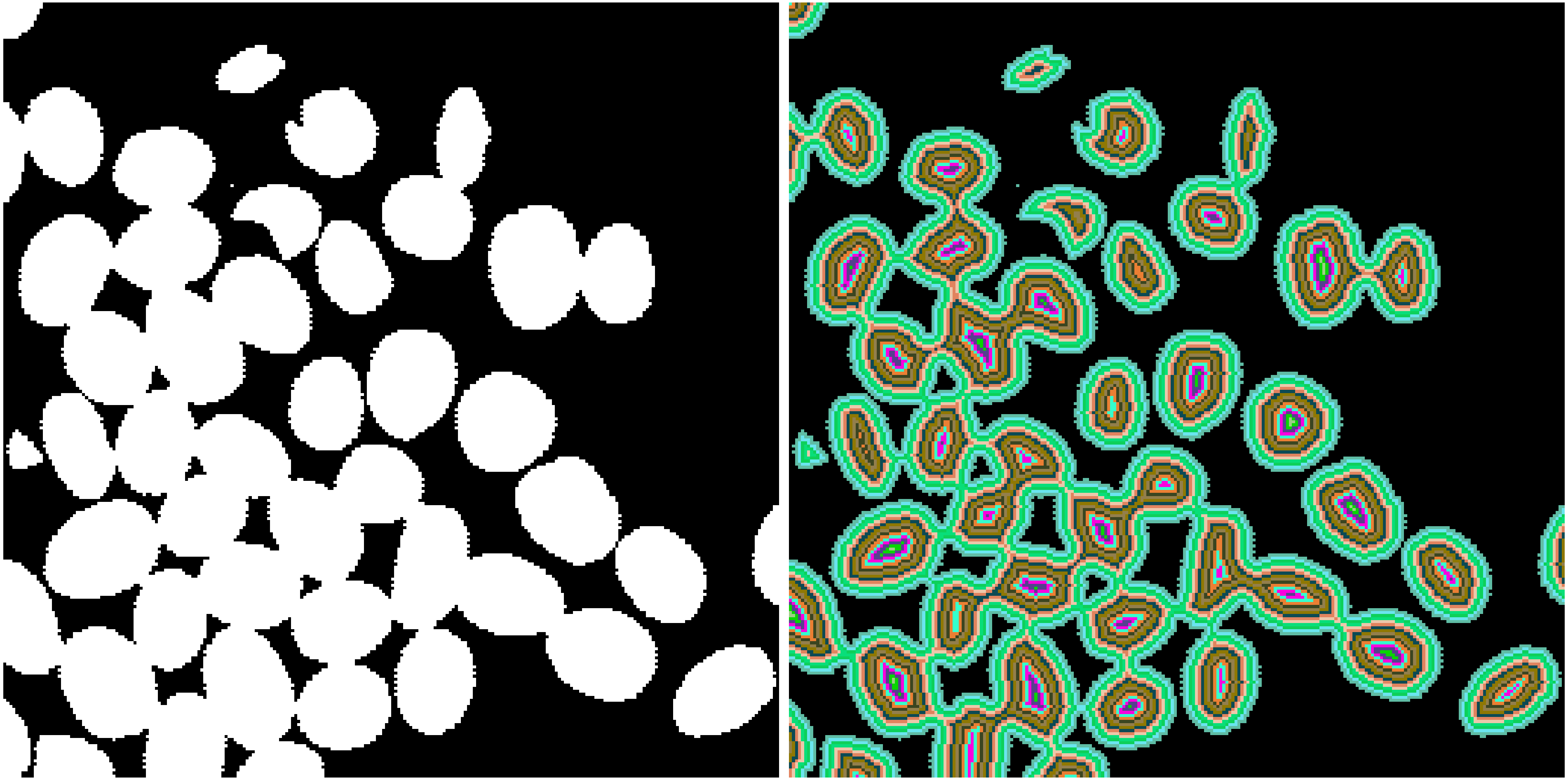


Binary opening

$$\boxed{X_B = D[E(X,B),\check{B}] \quad (or\ D[E(X,B),B])}$$

Binary opening

$$\boxed{\color{red}{X_B = D[E(X,B),\check{B}]} \quad (or\ D[E(X,B),B])}$$

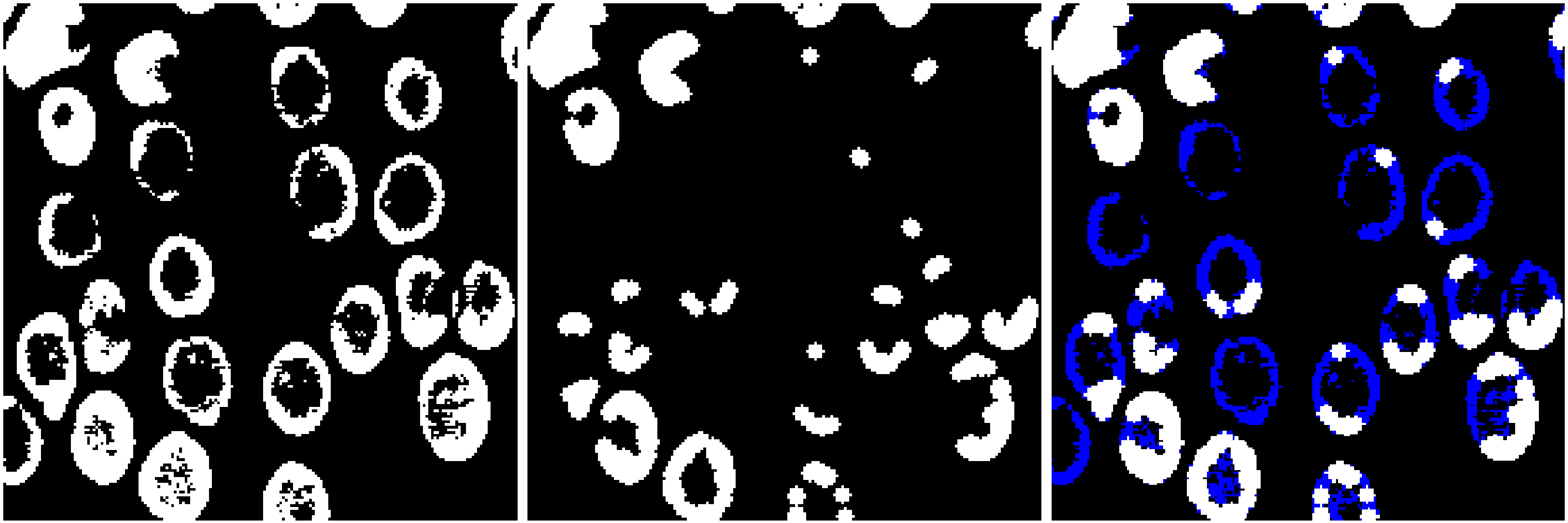
Properties of opening:

- anti-extensive ($X \supseteq X_B$);
- increasing ($X \subseteq Y \Rightarrow X_B \subseteq Y_B$);
- idempotent ($(X_B)_B = X_B$).

$\Rightarrow$ Morphological filter

- $B \subseteq B' \Rightarrow X_{B'} \subseteq X_B$;
- $(X_n)_{n'} = (X_{n'})_n = X_{\max(n,n')}$.

Example of opening



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Properties of closing:

- extensive ($X \subseteq X^B$);
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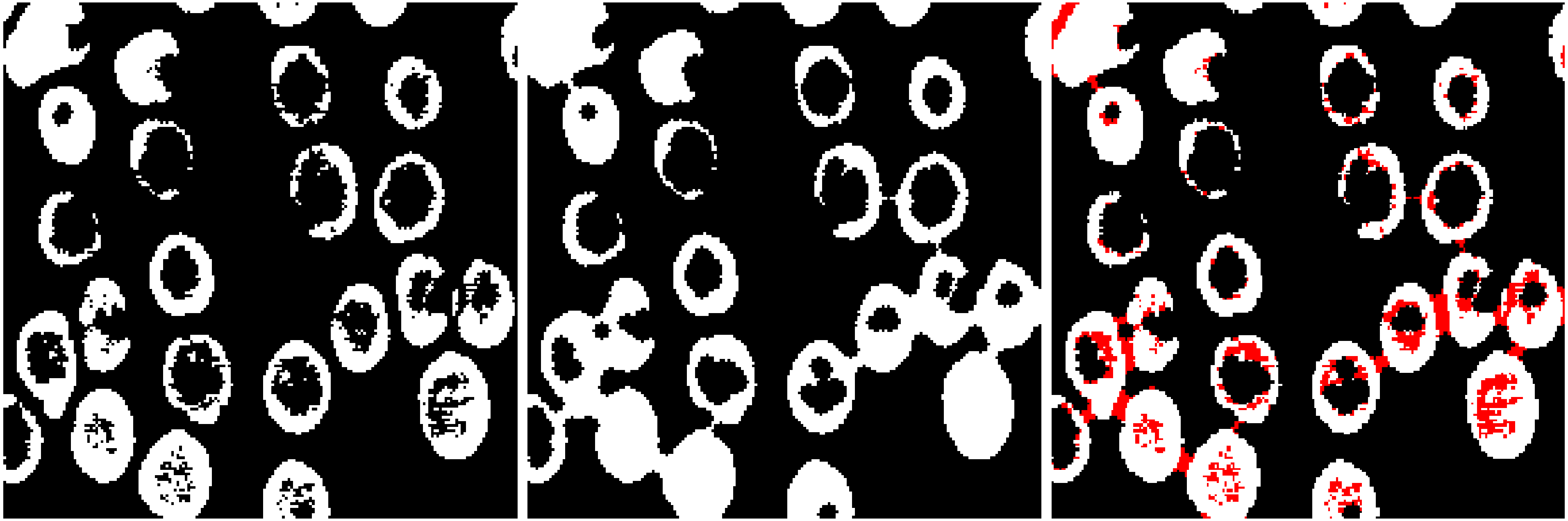
- idempotent ($(X^B)^B = X^B$).

$\Rightarrow$ Morphological filter

- $B \subseteq B' \Rightarrow X^B \subseteq X^{B'}$;
- $(X^n)^{n'} = (X^{n'})^n = X^{\max(n,n')}$;
- $X^B = [(X^C)_B]^C$.

Example of closing



Digital case

- choice of the digital grid (both for the image and the structuring element)
- translations on the grid
- same properties

From sets to functions

- subgraph of a function on $\mathbb{R}^n$ = subset of $\mathbb{R}^{n+1}$
- cuts of a function = sets

$$f_\lambda = \{x | f(x) \geq \lambda\}$$

$$D(f_\lambda, B) = [D(f, B)]_\lambda$$

- functional equivalents of set operations:

$$\begin{array}{ccc} \cup & \rightarrow & \sup / \vee \\ \cap & \rightarrow & \inf / \wedge \\ \subseteq & \rightarrow & \leq \\ \supseteq & \rightarrow & \geq \end{array}$$

Dilation of a function

by a flat structuring element

$$\boxed{\forall x \in \mathbb{R}^n, \quad D(f,B)(x) = \sup\{f(y) \ / \ y \in B_x\}}$$

Dilation of a function by a flat structuring element

$$\boxed{\forall x \in \mathbb{R}^n, \quad D(f, B)(x) = \sup\{f(y) \ / \ y \in B_x\}}$$

Properties of functional dilation:

- extensivity if $O \in B$;
- increasingness;
- $D(f \vee g, B) = D(f, B) \vee D(g, B)$;
- $D(f \wedge g, B) \leq D(f, B) \wedge D(g, B)$;
- iterativity property.

It holds:

$$D(f_\lambda, B) = [D(f, B)]_\lambda$$

Example of functional dilation

Erosion of a function

$$\boxed{\forall x \in \mathbb{R}^n, \quad E(f, B)(x) = \inf\{f(y) \ / \ y \in B_x\}}$$

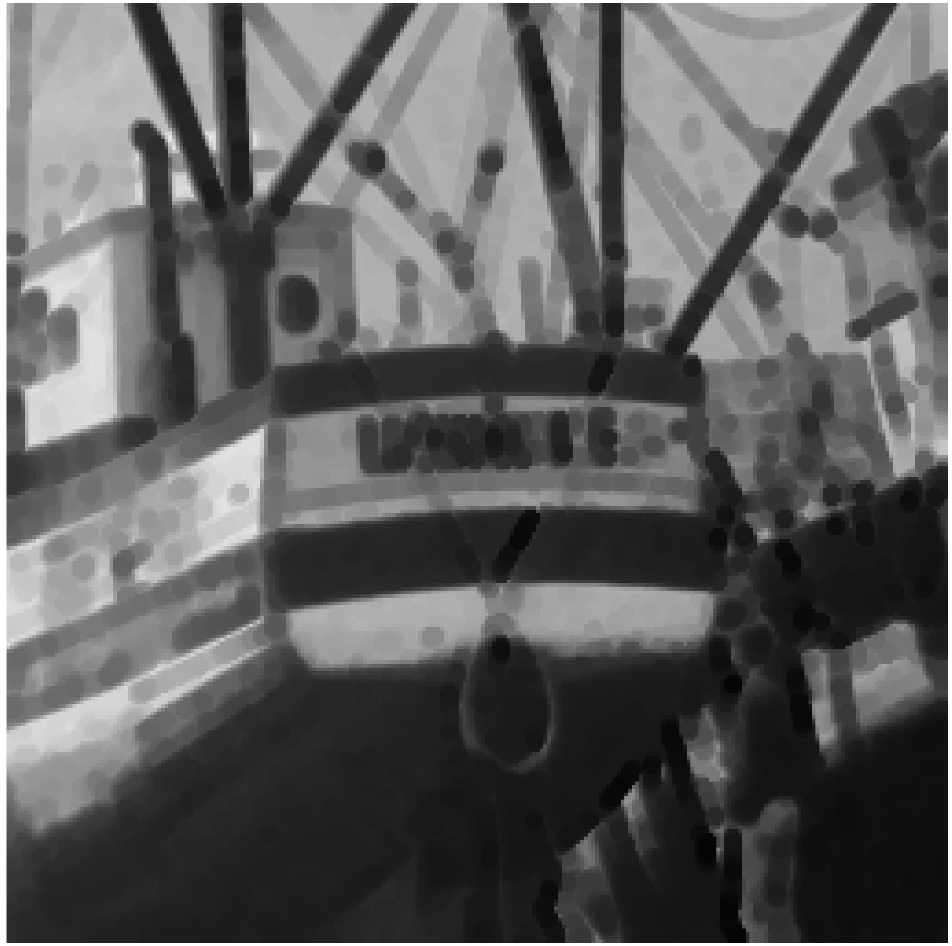
Erosion of a function

$$\boxed{\forall x \in \mathbb{R}^n, \quad E(f, B)(x) = \inf\{f(y) \;/\; y \in B_x\}}$$

Properties of functional erosion:

- functional dilation and erosion are dual operators;
- anti-extensivity if $O \in B$;
- increasingness;
- $E(f \vee g, B) \geq E(f, B) \vee E(g, B)$;
- $E(f \wedge g, B) = E(f, B) \wedge E(g, B)$;
- iterativity property.

Example of functional erosion



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Example of functional opening

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Properties of functional closing:

- extensive;
- increasing;
- idempotent.

$\Rightarrow$ morphological filter

- duality between opening and closing

Example of functional closing

Structuring functions

Dilation:

$$D(f,g)(x) = \sup_y \{f(y) + g(y-x)\}$$

Erosion:

$$E(f,g)(x) = \inf_y \{f(y) - g(y-x)\}$$

Flat structuring element:

$$g(x) = \begin{cases} 0 & \text{on a compact } B \\ -\infty & \text{elsewhere} \end{cases}$$

Some applications of erosion and dilation

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Contrast enhancement

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Contrast enhancement: ES 15, $\alpha = \beta = 0.2$, $\alpha = \beta = 0.3$, $\alpha = \beta = 0.5$

Some applications of erosion and dilation

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Some applications of erosion and dilation

Morphological gradient: $\boxed{D_B(x) - E_B(x)}$

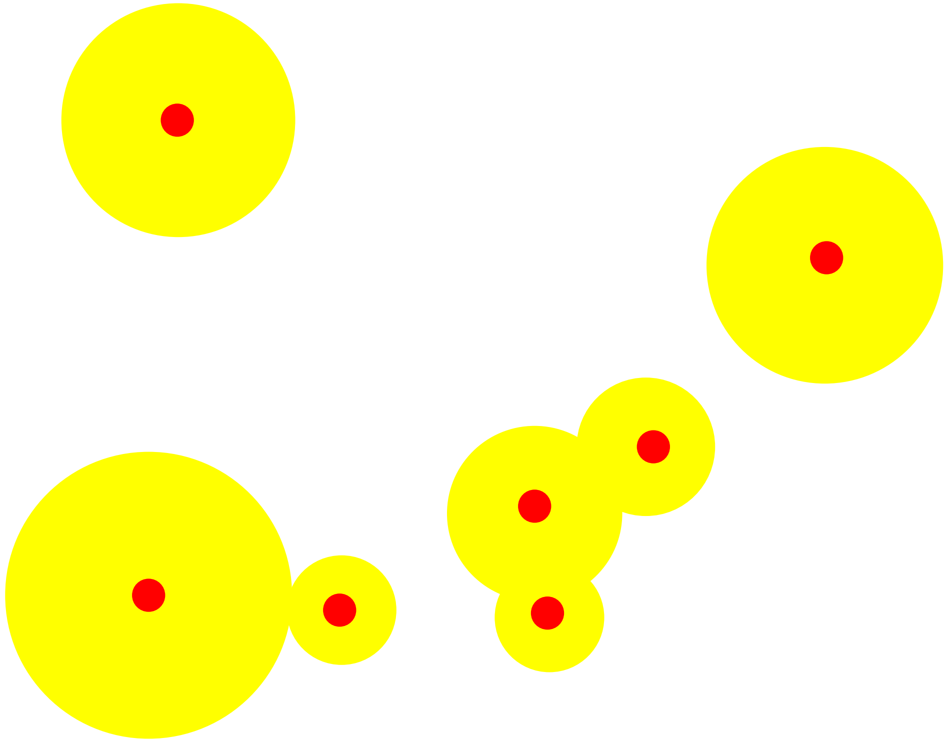
Some applications of erosion and dilation

Ultimate erosion:

$$EU(X) = \cup_n \{E(X, B_n) \setminus R[E(X, B_{n+1}); E(X, B_n)]\}$$

- $E(X, B_n)$: erosion of X by a structuring element of size n
- $R[Y; Z]$: connected components of Z having a non-empty intersection with Y

= set of regional maxima of the distance function $d(x, X^C)$.



An application of opening: top-hat transform

$$\boxed{f - f_B}$$

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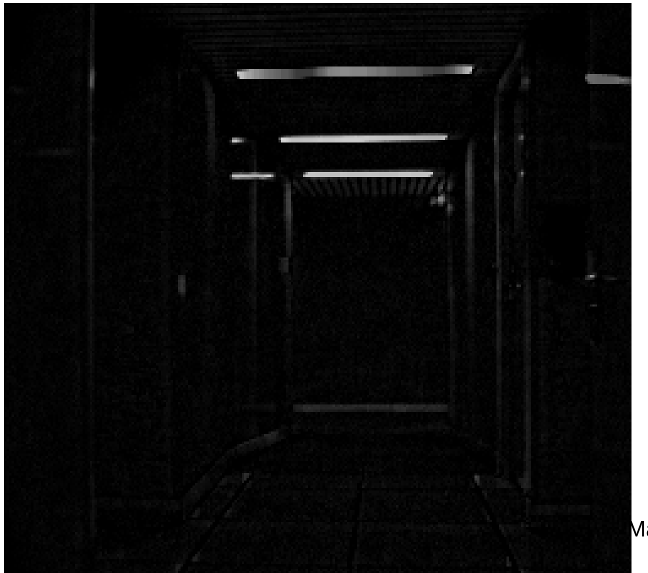

Granulometry

- $\forall X \in \mathcal{A}, \forall \lambda > 0, \;\; \phi_\lambda(X) \subseteq X$ (ϕ_λ anti-extensive);
- $\forall (X, Y) \in \mathcal{A}^2, \forall \lambda > 0, \;\; X \subseteq Y \Rightarrow \phi_\lambda(X) \subseteq \phi_\lambda(Y)$ (ϕ_λ increasing);
- $\forall X \in \mathcal{A}, \forall \lambda > 0, \forall \mu > 0 \;\; \lambda \geq \mu \Rightarrow \phi_\lambda(X) \subseteq \phi_\mu(X)$ (ϕ_λ decreasing with respect to the parameter);
- $\forall \lambda > 0, \forall \mu > 0, \;\; \phi_\lambda \circ \phi_\mu = \phi_\mu \circ \phi_\lambda = \phi_{\max(\lambda, \mu)}$.

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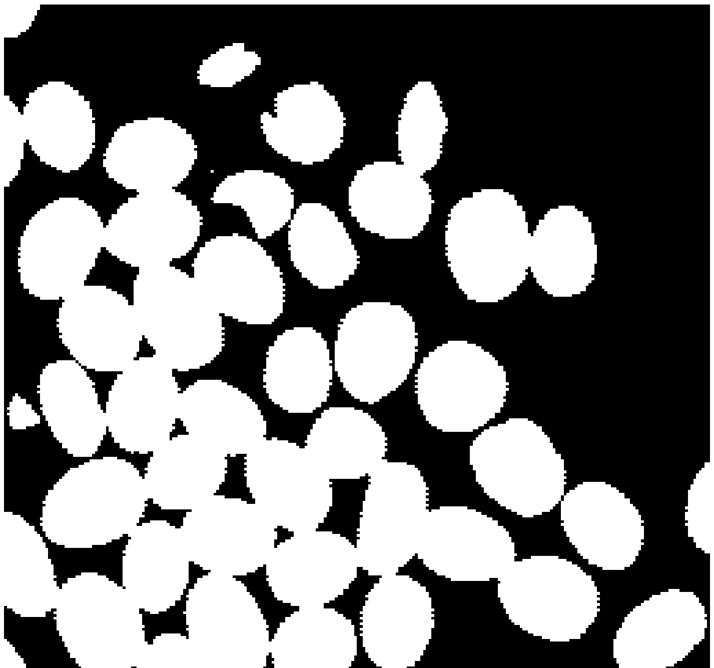
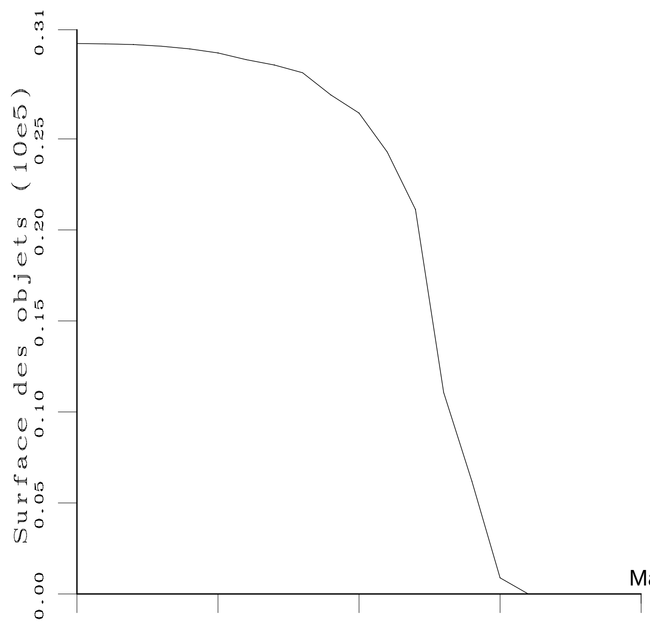
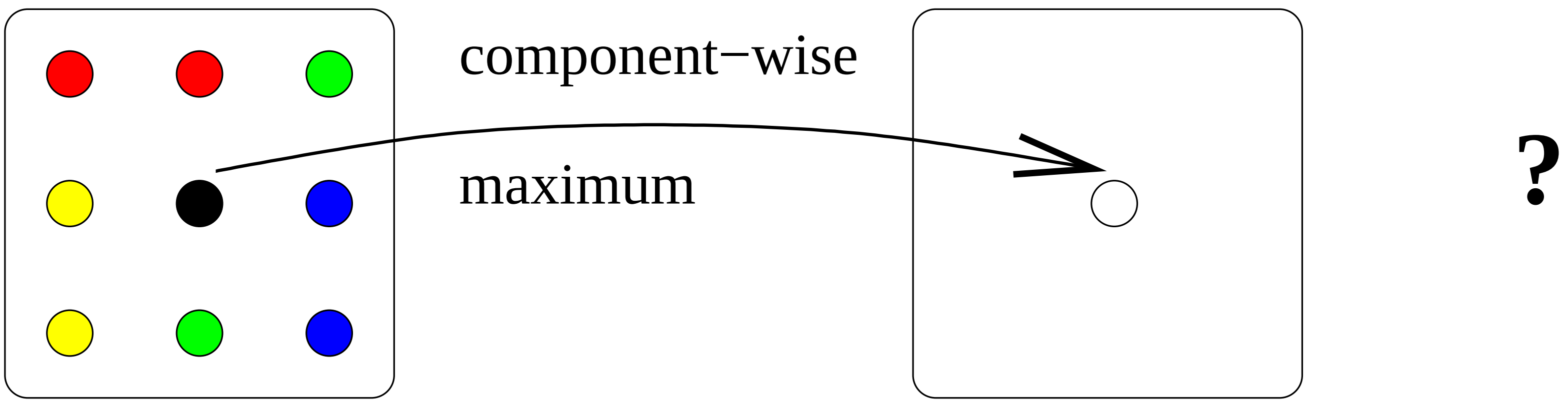




Vectorial functions (e.g. color images)

- Main difficulty: choice of an ordering
- component-wise max (or min): no good properties

Dilation



Choice of the structuring element

- depends on what one wants suppress / keep
- shape
- size

Example: opening by disks or segments?

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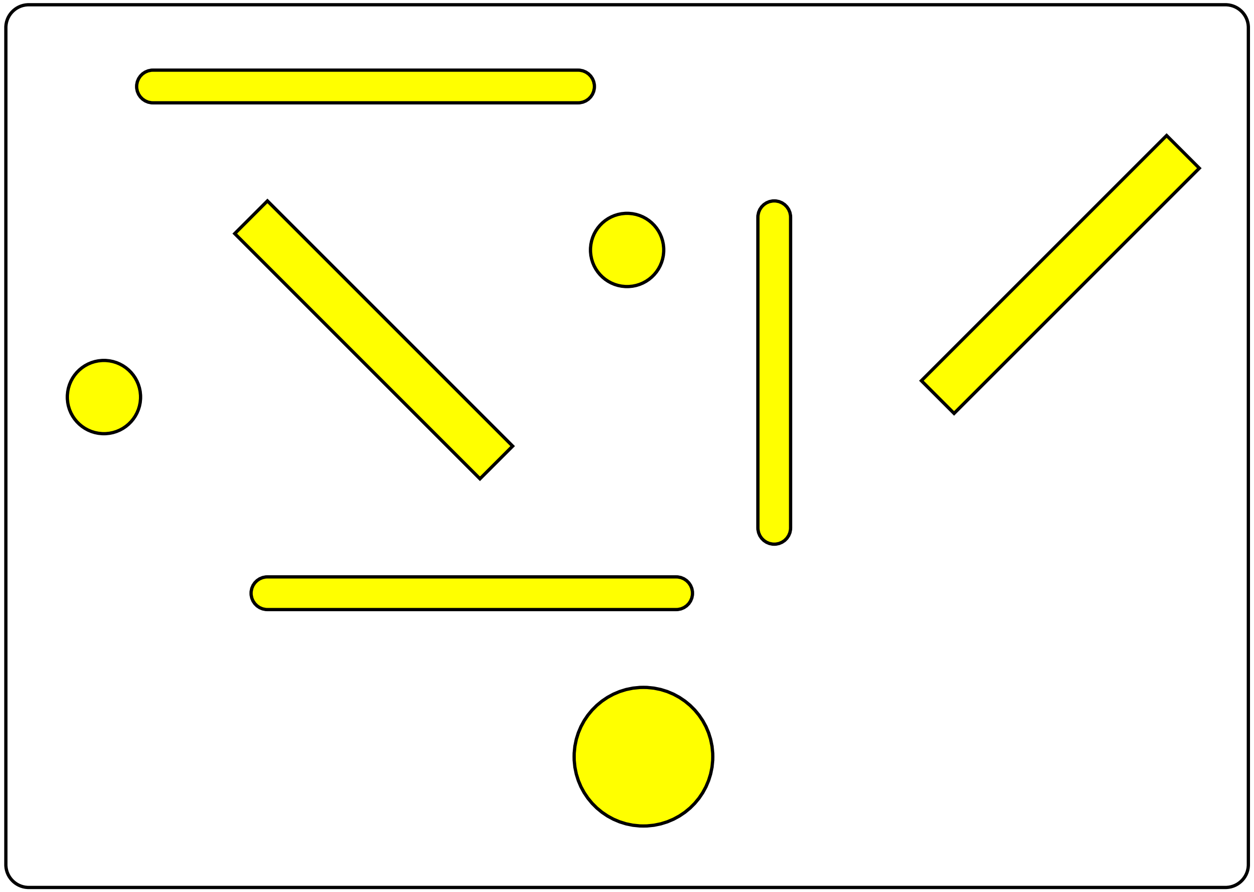

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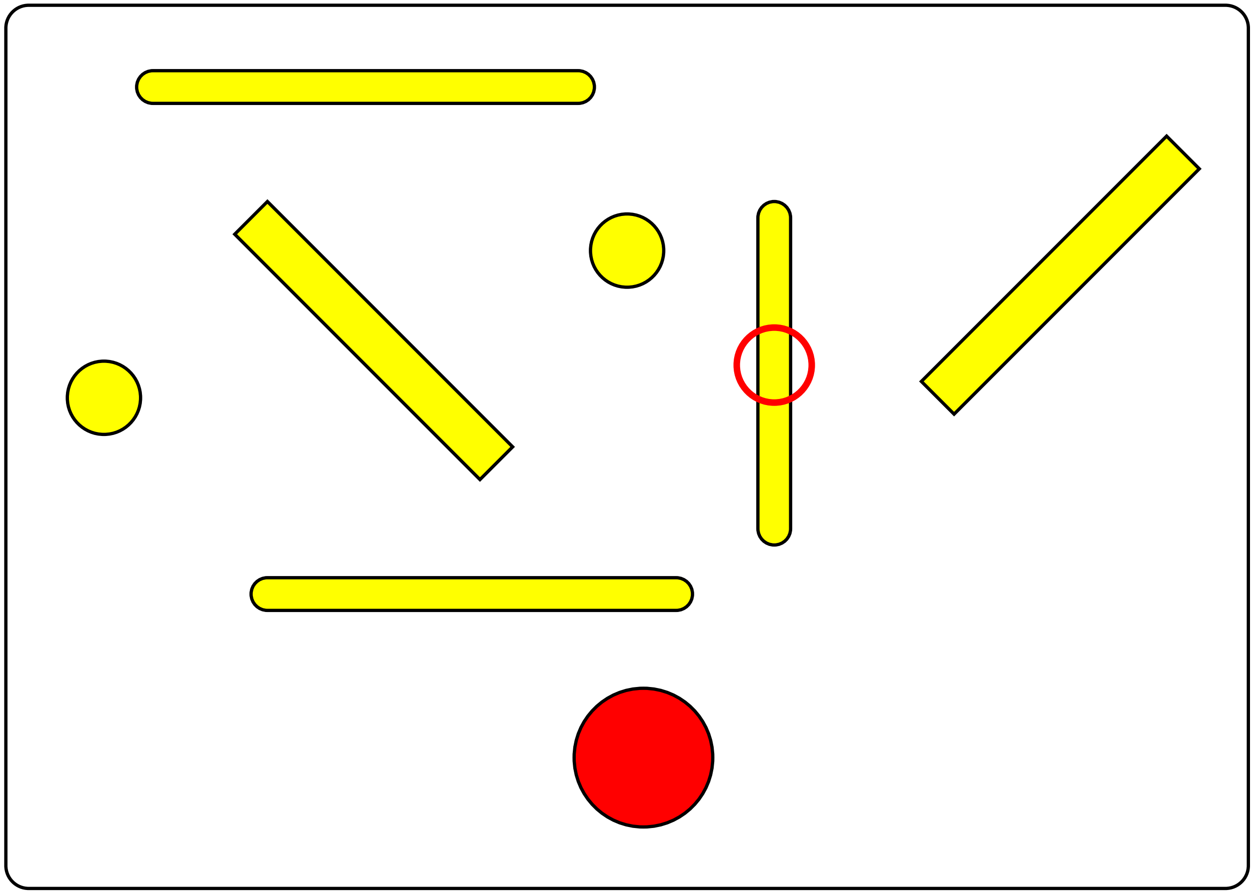

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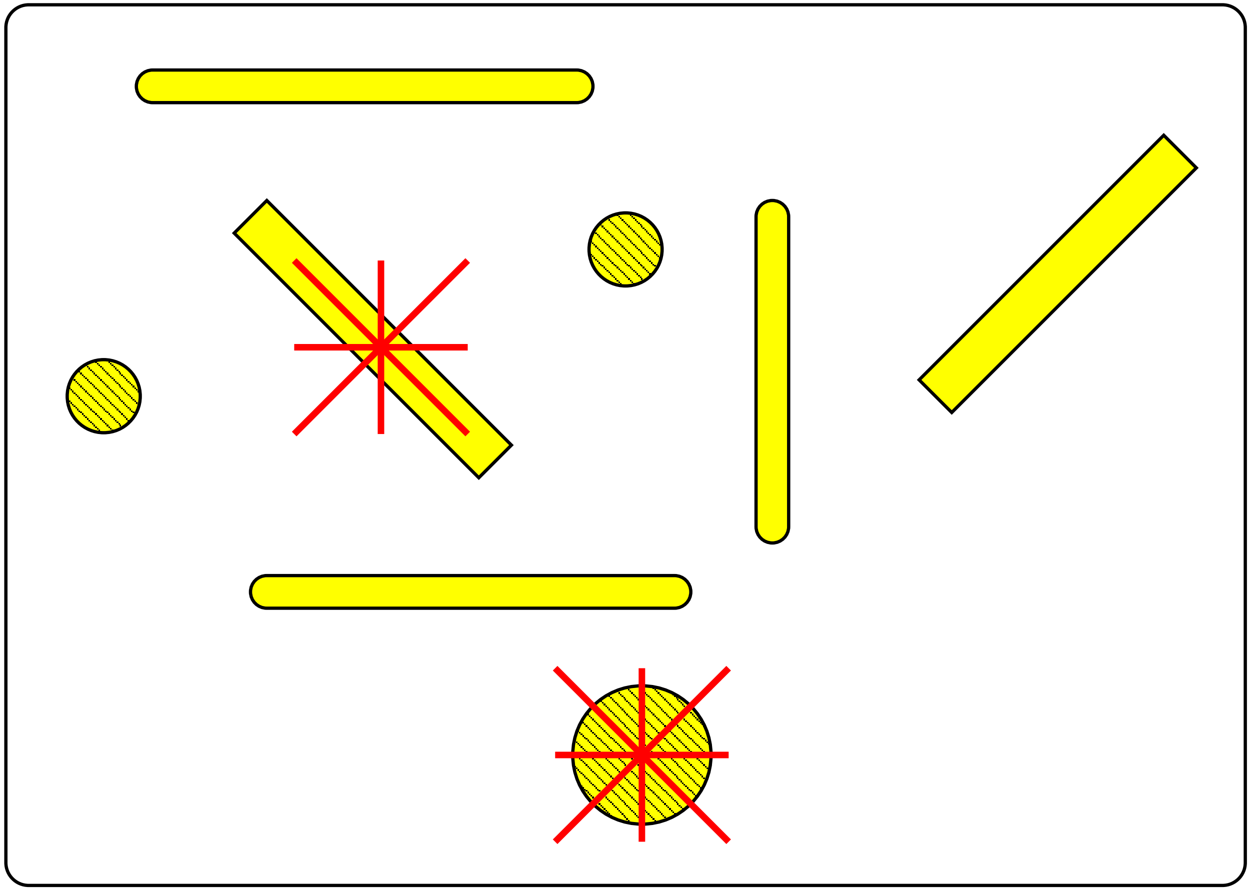

Rq: a union of openings is an opening

Surfacic opening

$$\gamma_\lambda(f) = \bigvee_i \{\gamma_{B_i}(f),\ B_i \text{ connected and } S(B_i) = \lambda\}$$

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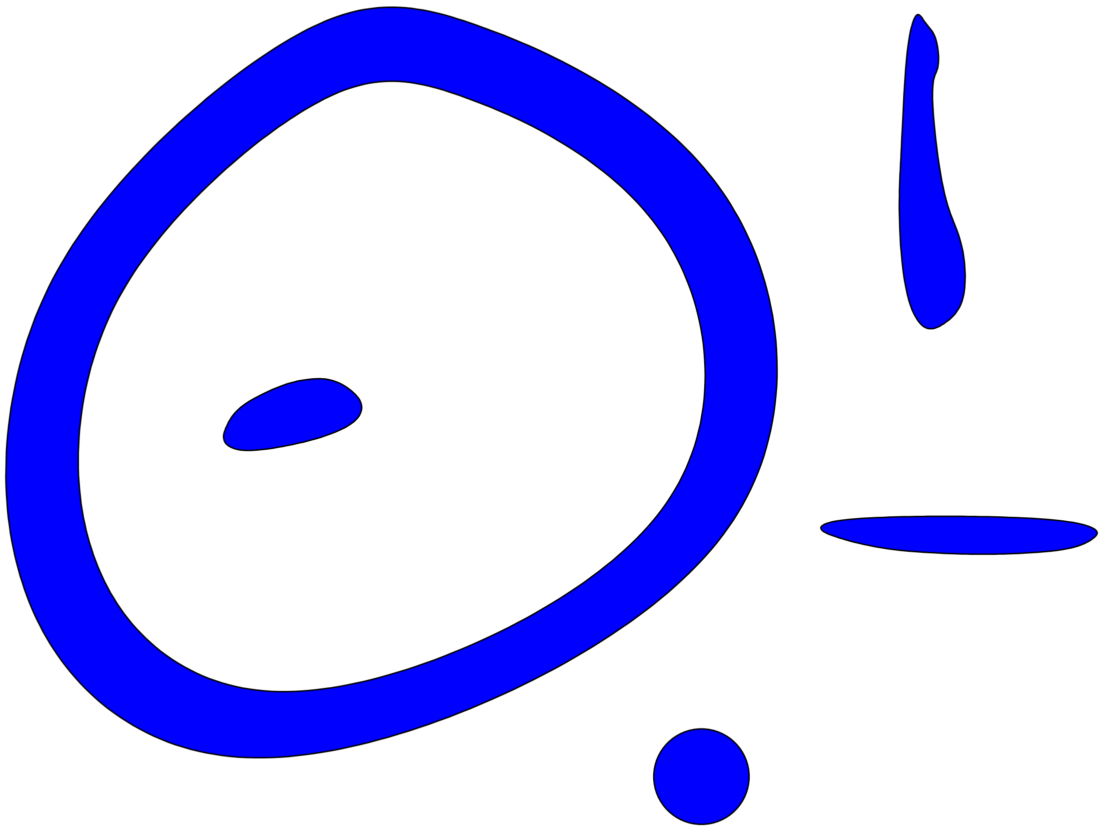

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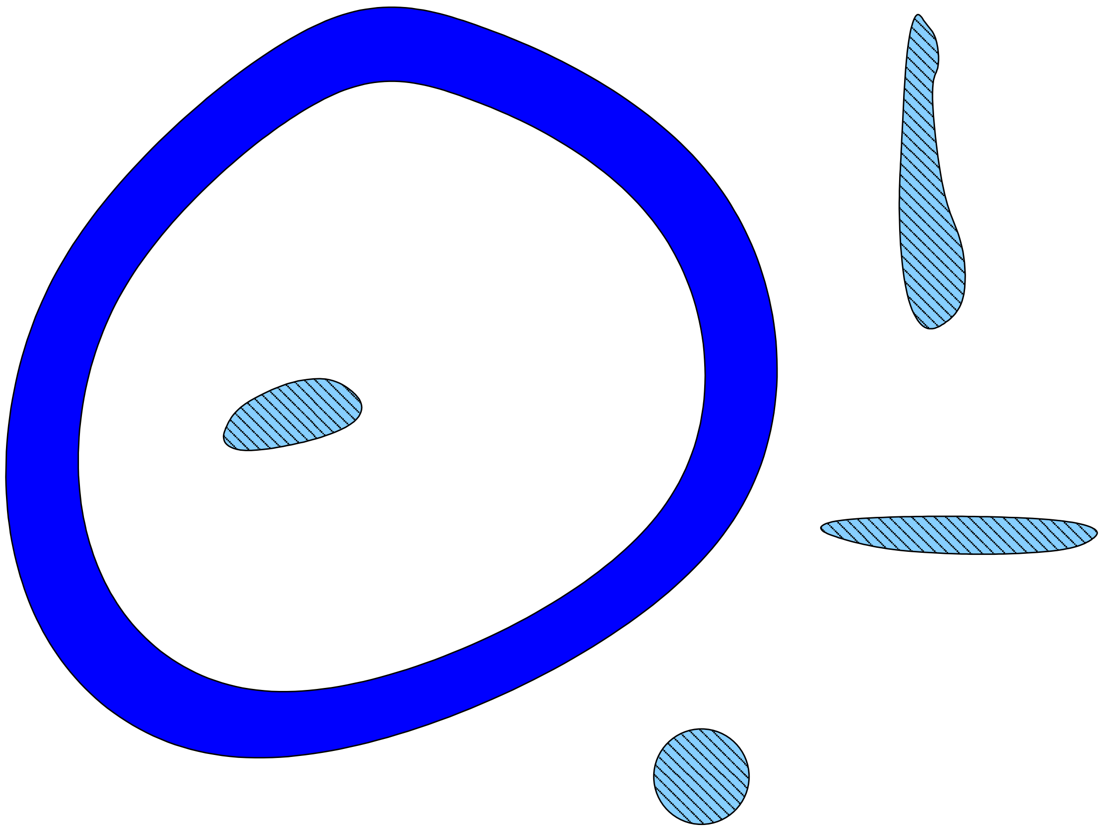

Mathematical fundaments of mathematical morphology

- Set theory
  - relations ($\subseteq$, $\cap$, $\cup$...)
  - structuring element
- Topology
  - hit-or-miss topology (Fell's topology)
  - myopic topology
  - Hausdorff distance
- Lattice theory
  - adjunctions
  - algebraic operations
- Probability theory
  - $P(A \cap K \neq \emptyset)$
  - random closed sets

Hit-or-miss topology

- topology on closed subsets
- generated by $\mathcal{F}^K$ and $\mathcal{F}_G$ (K compact and G open):

$$\mathcal{F}^K = \{F \in \mathcal{F}, F \cap K = \emptyset\}$$

$$\mathcal{F}_G = \{F \in \mathcal{F}, F \cap G \neq \emptyset\}$$

- convergence in $\mathcal{F}$: $(F_n)_{n \in \mathbb{N}}$ converges towards $F \in \mathcal{F}$ if:

$$\begin{cases} \forall G \in \mathcal{G}, G \cap F \neq \emptyset, \exists N, \forall n \geq N, \quad G \cap F_n \neq \emptyset \\ \forall K \in \mathcal{K}, K \cap F = \emptyset, \exists N', \forall n \geq N', \; K \cap F_n = \emptyset \end{cases}$$

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Union is continuous from $\mathcal{F} \times \mathcal{F}$ in $\mathcal{F}$ but intersection is not

$\Downarrow$

semi-continuity

Semi-continuity

$f : \Omega \rightarrow \mathcal{F}$

- f upper semi-continuous (u.s.c.) if $\forall \omega \in \Omega$ and $\forall (\omega_n)_{n \in \mathbb{N}} \in \Omega$ converging towards ω :

$$\overline{lim} f(\omega_n) \subseteq f(\omega)$$

- f lower semi-continuous (l.s.c.) if:

$$\underline{lim} f(\omega_n) \supseteq f(\omega)$$

$\overline{lim}/\underline{lim}$ = $\cup/\cap$ of adherence points

$\boxed{f \text{ continuous iff } f \text{ l.s.c. and u.s.c.}}$

Intersection is u.s.c.

Properties of morphological operations

- the dilation of a closed set by a compact set is continuous
- the dilation of a compact set by a compact set is continuous
- $(F, K) \mapsto E(F, K)$ u.s.c.
- $(K', K) \mapsto E(K', K)$ u.s.c.
- $(F, K) \mapsto F_K$ u.s.c.
- $(K', K) \mapsto K'_K$ u.s.c.
- $(F, K) \mapsto F^K$ u.s.c.
- $(K', K) \mapsto K'^K$ u.s.c.

Myopic topology

- generated by:

$$\mathcal{K}_G^F = \{K \in \mathcal{K}, K \cap F = \emptyset, K \cap G \neq \emptyset\}$$

  ($F \in \mathcal{F}$, $G \in \mathcal{G}$)
- finer than the topology induced on $\mathcal{K}$ by the hit-or-miss topology
- equivalent on $\mathcal{K} \setminus \emptyset$ to the topology induced by the Hausdorff distance

$$\delta(K, K') = \max\{\sup_{x \in K} d(x, K'), \sup_{x' \in K'} d(x', K)\}$$

Rq: $\delta(K, K') = \inf\{\varepsilon,\ K \subseteq D(K', B^\varepsilon), K' \subseteq D(K, B^\varepsilon)\}$

Algebraic framework: complete lattices

- Lattice: $(\mathcal{T}, \leq)$ ($\leq$ ordering) such that $\forall (x,y) \in \mathcal{T}, \exists x \vee y$ and $\exists x \wedge y$
- Complete lattice: every family of elements (finite or not) has a smallest upper bound and a largest lower bound
- $\Rightarrow$ contains a smallest element 0 and a largest element I:

$$0 = \bigwedge \mathcal{T} = \bigvee \emptyset \text{ et } I = \bigvee \mathcal{T} = \bigwedge \emptyset$$

- Examples of complete lattices:
  - $(\mathcal{P}(E), \subseteq)$: complete lattice, Boolean (complemented and distributive):

$$\forall x, \exists x^C, x \wedge x^C = 0 \text{ and } x \vee x^C = I$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

  - $(\mathcal{F}(\mathbb{R}^d), \subseteq)$
  - functions of $\mathbb{R}^n$ in $\overline{\mathbb{R}}$ for the ordering $\leq$:

$$f \leq g \Leftrightarrow \forall x \in \mathbb{R}^n, \quad f(x) \leq g(x)$$

  - partitions

Semi-continuity of functions

- u.s.c. :

$$\forall t > f(x), \exists V(x), \forall y \in V(x), \quad t > f(y)$$

  ($V(x)$ neighborhood of x in $\mathbb{R}^n$)

- l.s.c. :

$$\forall t < f(x), \exists V(x), \forall y \in V(x), t < f(y)$$

- a function is u.s.c. iff its sub-graph is closed
- topology on the space of u.s.c. functions = topology induced by the hit-or-miss topology on $\mathcal{F}(\mathbb{R}^n \times \overline{\mathbb{R}})$
- the set of u.s.c. functions of $\mathbb{R}^n$ in $\overline{\mathbb{R}}$ is a complete lattice for $\leq$:

$$f \leq g \Leftrightarrow SG(f) \subseteq SG(g)$$

Algebraic dilation and erosion

complete lattice $(\mathcal{T}, \leq)$

Algebraic dilation:

$$\forall (x_i) \in \mathcal{T},\ \delta(\vee_i x_i) = \vee_i \delta(x_i)$$

Algebraic erosion:

$$\forall (x_i) \in \mathcal{T},\ \varepsilon(\wedge_i x_i) = \wedge_i \varepsilon(x_i)$$

Properties:

- $\delta(0) = 0$ (in $\mathcal{P}(E), 0 = \emptyset$)
- $\varepsilon(I) = I$ (in $\mathcal{P}(E), I = E$)
- δ increasing
- ε increasing
- in $\mathcal{P}(\mathbb{R}^n)$, $\delta(X) = \cup_{x \in X} \delta(\{x\})$

Adjunctions

(ε, δ) adjunction on $(\mathcal{T}, \leq)$:

$$\forall (x, y),\ \delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y)$$

Properties:

- $\delta(0) = 0$ and $\varepsilon(I) = I$
- (ε, δ) adjunction $\Rightarrow$ ε = algebraic erosion and δ = algebraic dilation
- δ increasing = algebraic dilation iff $\exists\varepsilon$ such that (ε, δ) is an adjunction
  $\Rightarrow$ ε = algebraic erosion and $\varepsilon(x) = \bigvee\{y \in \mathcal{T},\ \delta(y) \leq x\}$
- ε increasing = algebraic erosion iff $\exists\delta$ such that (ε, δ) is an adjunction
  $\Rightarrow$ δ = algebraic dilation and $\delta(x) = \bigwedge\{y \in \mathcal{T},\ \varepsilon(y) \geq x\}$
- $\varepsilon\delta \geq Id$
- $\delta\varepsilon \leq Id$
- $\varepsilon\delta\varepsilon = \varepsilon$
- $\delta\varepsilon\delta = \delta$
- $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$ and $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$

Links with morphological operators

- On the lattice of the subsets of $\mathbb{R}^n$ or $\mathbb{Z}^n$, with inclusion:

$$\delta(X) = \cup_{x \in X} \delta(\{x\})$$

- + invariance under translation $\Rightarrow \exists B,\ \delta(X) = D(X, B)$
- Same result on the lattice of functions.
- Similar results for erosion.

Algebraic opening and closing

- Algebraic opening: γ increasing, idempotent and anti-extensive
- Algebraic closing: φ increasing, idempotent and extensive
- Examples: $\gamma = \delta\varepsilon$ and $\varphi = \varepsilon\delta$ with (ε, δ) = adjunction
- Invariance domain: $Inv(\varphi) = \{x \in \mathcal{T},\ \varphi(x) = x\}$
- γ opening $\Rightarrow \gamma(x) = \bigvee\{y \in Inv(\gamma),\ y \leq x\}$
- φ closing $\Rightarrow \varphi(x) = \bigwedge\{y \in Inv(\varphi),\ x \leq y\}$
- (γ_i) openings $\Rightarrow \bigvee_i \gamma_i$ opening
- (φ_i) closings $\Rightarrow \bigwedge_i \varphi_i$ closing
- γ_1 and γ_2 openings $\Rightarrow$ equivalence between:
  1. $\gamma_1 \leq \gamma_2$
  2. $\gamma_1\gamma_2 = \gamma_2\gamma_1 = \gamma_1$
  3. $Inv(\gamma_1) \subseteq Inv(\gamma_2)$
- φ_1 and φ_2 closings $\Rightarrow$ equivalence between:
  1. $\varphi_2 \leq \varphi_1$
  2. $\varphi_1\varphi_2 = \varphi_2\varphi_1 = \varphi_1$
  3. $Inv(\varphi_1) \subseteq Inv(\varphi_2)$

Algebraic filter theory

Filter = increasing and idempotent operator

Examples

- openings γ and $\bigvee_i \gamma_i$ (anti-extensive filters)
- closings φ and $\bigwedge_i \varphi_i$ (extensive filters)

Theorem on filter composition φ and ψ such that $\varphi \geq \psi$:

- $\varphi \geq \varphi\psi\varphi \geq \varphi\psi \vee \psi\varphi \geq \varphi\psi \wedge \psi\varphi \geq \psi\varphi\psi \geq \psi$
- $\varphi\psi$, $\psi\varphi$, $\varphi\psi\varphi$ and $\psi\varphi\psi$ are filters
- $Inv(\varphi\psi\varphi) = Inv(\varphi\psi)$ and $Inv(\psi\varphi\psi) = Inv(\psi\varphi)$
- $\varphi\psi\varphi$ is the smallest filter which is largest than $\varphi\psi \vee \psi\varphi$

Example: alternate sequential filters

- openings γ_i and closings φ_i such that:

$$i \leq j \Rightarrow \gamma_j \leq \gamma_i \leq Id \leq \varphi_i \leq \varphi_j$$

- Theorem on filter composition $\Rightarrow m_i = \gamma_i \varphi_i$, $n_i = \varphi_i \gamma_i$, $r_i = \varphi_i \gamma_i \varphi_i$ and $s_i = \gamma_i \varphi_i \gamma_i$ are filters
- Alternate sequential filters:

$$\begin{aligned} M_i &= m_i m_{i-1} ... m_2 m_1 \\ N_i &= n_i n_{i-1} ... n_2 n_1 \\ R_i &= r_i r_{i-1} ... r_2 r_1 \\ S_i &= s_i s_{i-1} ... s_2 s_1 \end{aligned}$$

- Property: $i \leq j \Rightarrow M_j M_i = M_j, \; N_j N_i = N_j, ...$

Morphological alternate sequential filters

$$\boxed{(...(((f_{B_1})^{B_1})_{B_2})^{B_2})..._{B_n})^{B_n}}$$

Morphological alternate sequential filters

$$\boxed{(\ldots(((f_{B_1})^{B_1})_{B_2})^{B_2})\ldots_{B_n})^{B_n}}$$

Morphological alternate sequential filters

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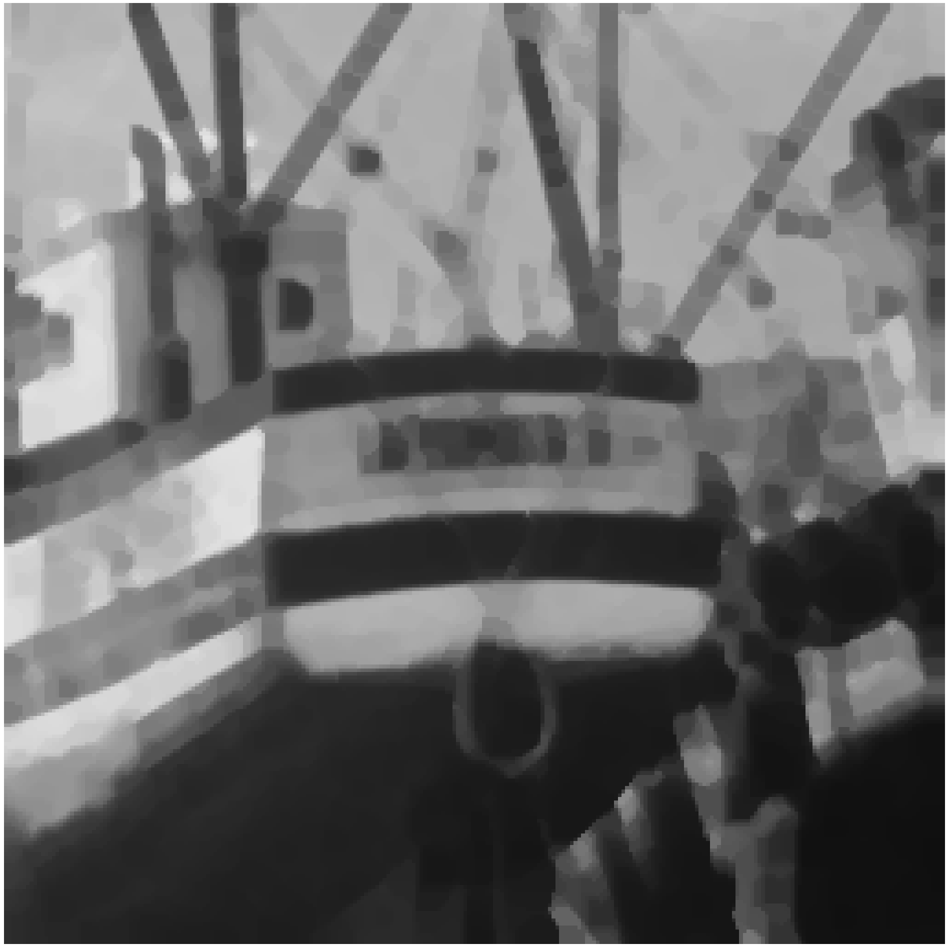

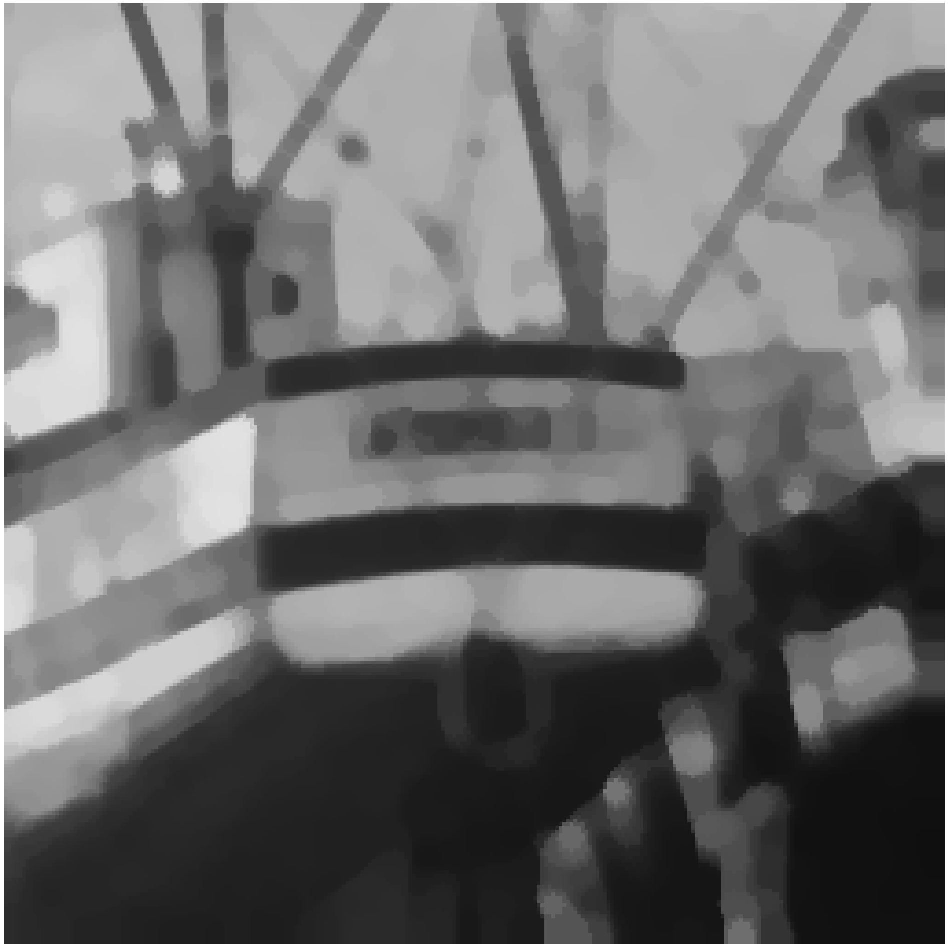

Auto-dual filters

- Operators which are independent of the local contrast, acting similarly on bright and dark areas.
- Example: morphological center

$$Median[f(x), \psi_1(f)(x), \psi_2(f)(x)]$$

- More generally, for operators $\{\psi_1, \psi_2, ...\psi_n\}$: $(Id \vee \wedge_i \psi_i) \wedge \vee_i \psi_i$
- For instance $\psi_1(f) = \gamma\varphi(f) = (f^B)_B$, $\psi_2 = \varphi\gamma(f) = (f_B)^B$

Auto-dual filters

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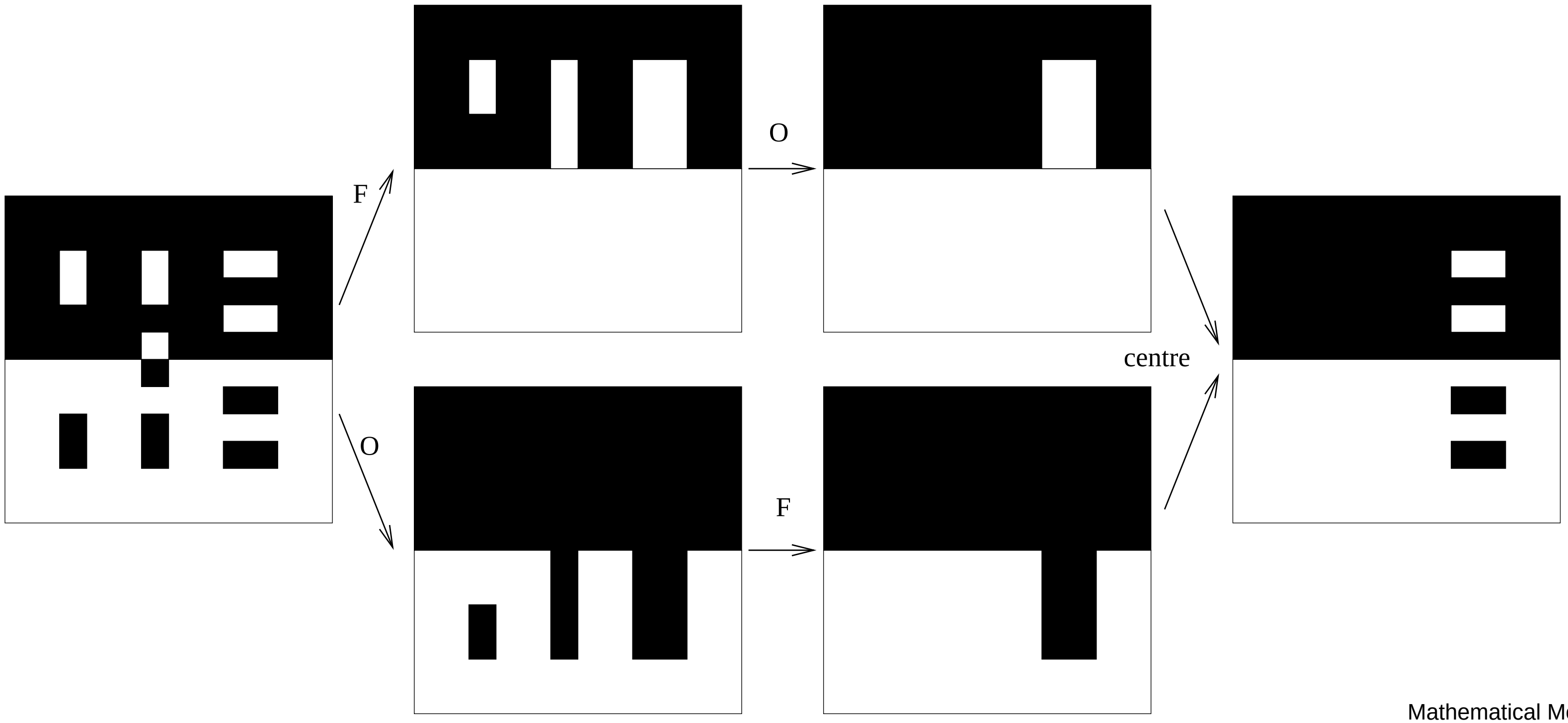
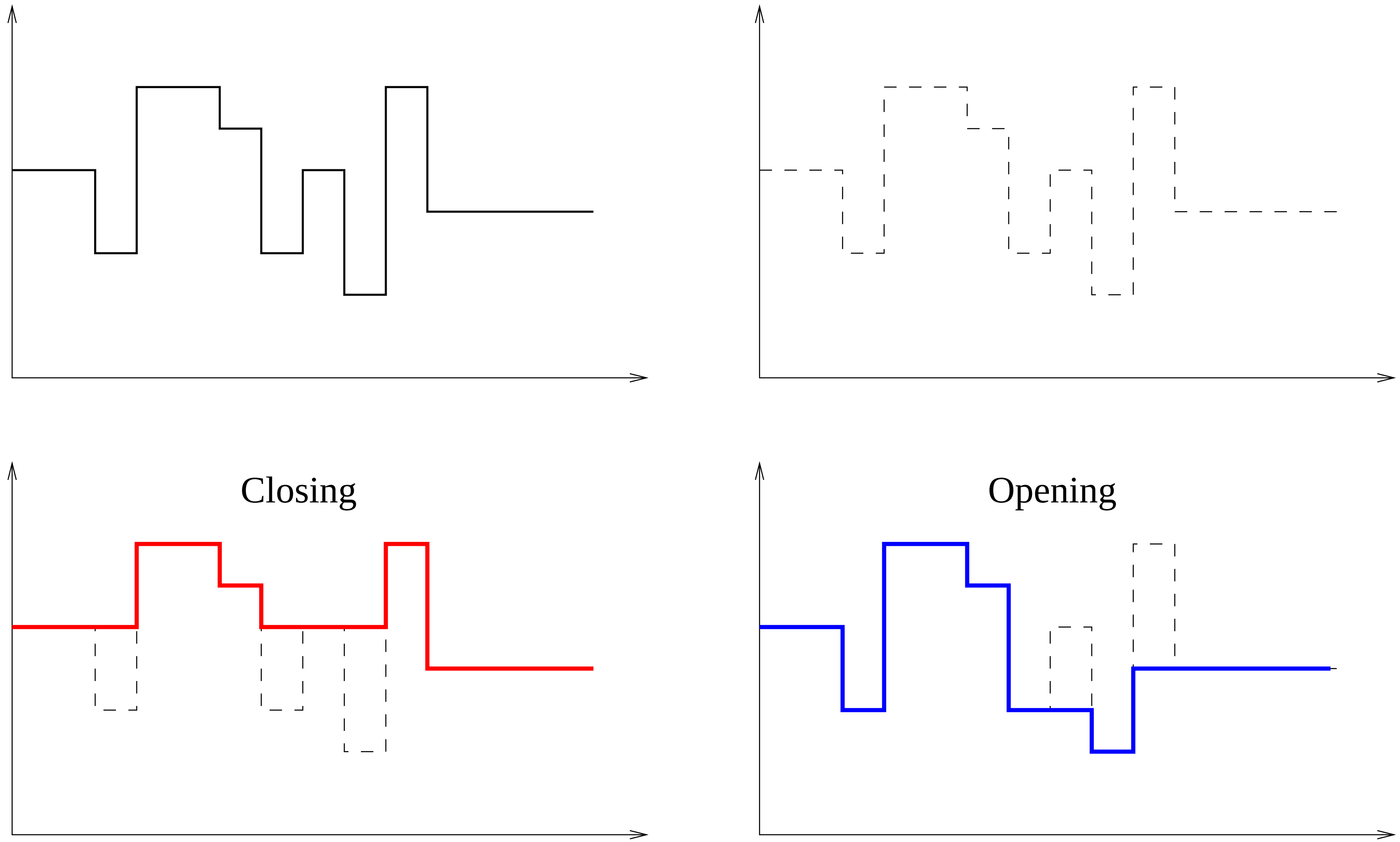


Morphological center: numerical example



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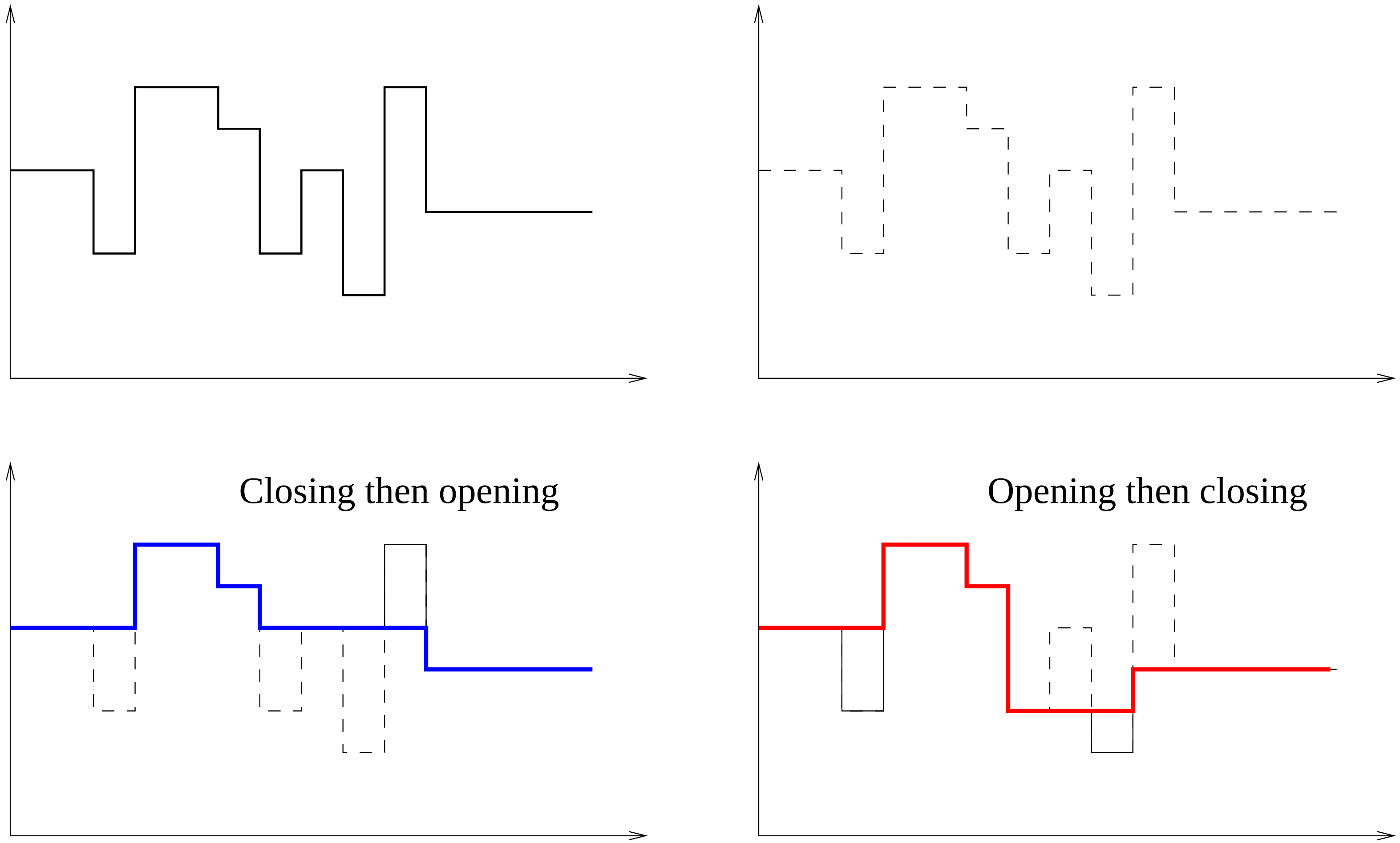

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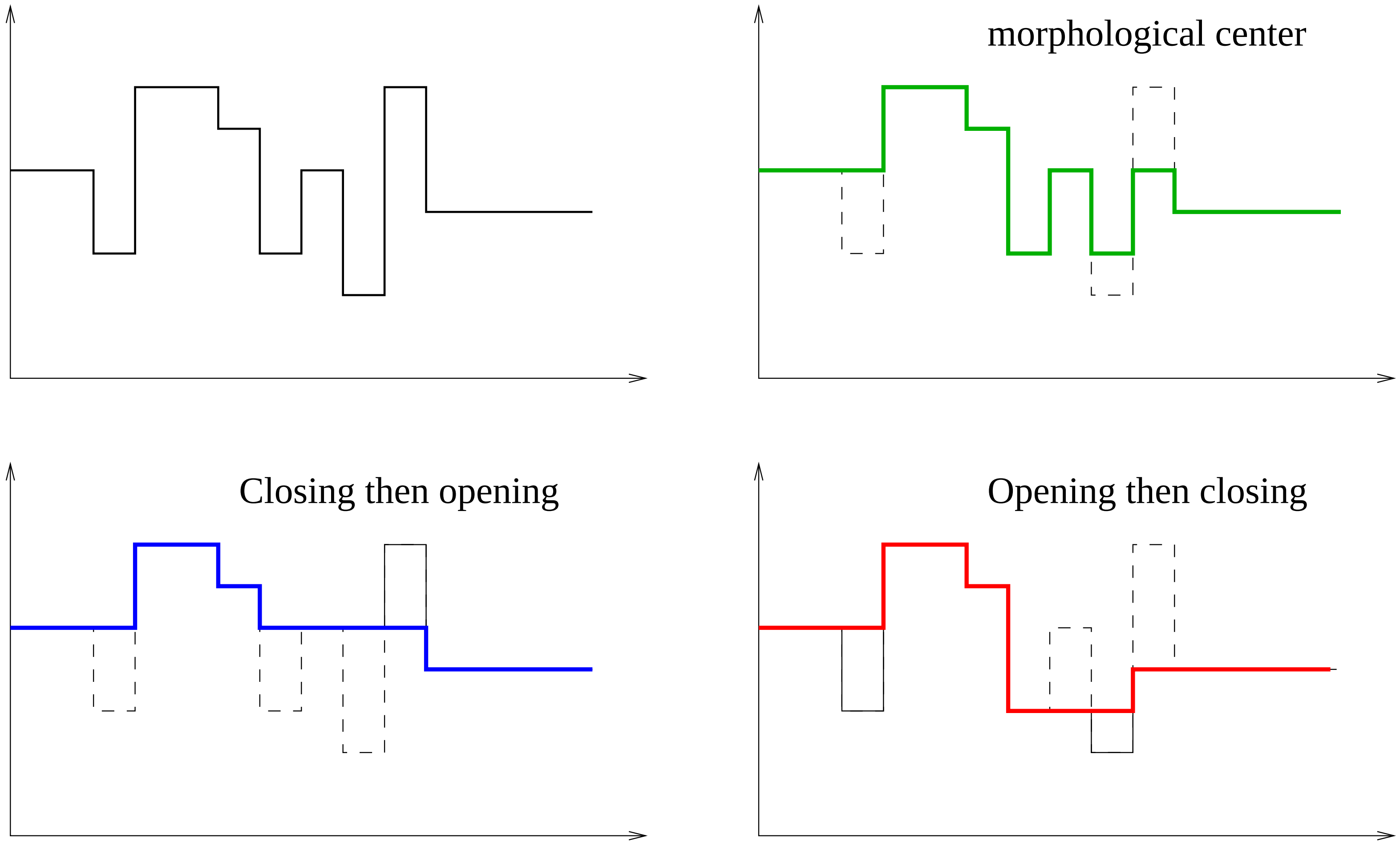


Hit-or-Miss Transformation

Structuring element: $T = (T_1, T_2)$, with $T_1 \cap T_2 = \emptyset$

HMT:

$$X \otimes T = E(X, T_1) \cap E(X^C, T_2)$$

Hit-or-Miss Transformation

Structuring element: $T = (T_1, T_2)$, with $T_1 \cap T_2 = \emptyset$

HMT:

$$X \otimes T = E(X, T_1) \cap E(X^C, T_2)$$

Thinning (if $O \in T_1$):

$$X \circ T = X \setminus X \otimes T$$

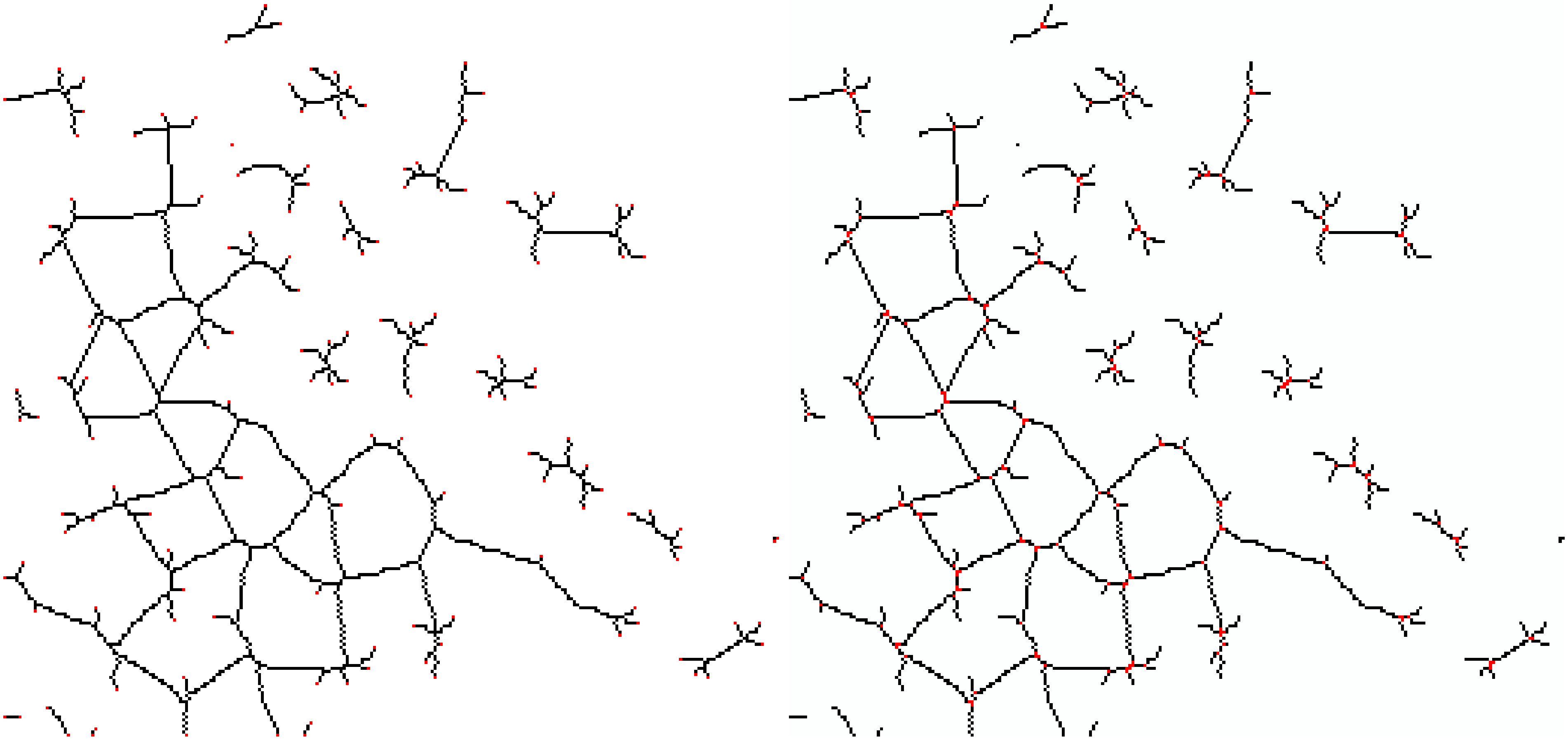
Thickening (if $O \in T_2$):

$$X \odot T = X \cup X \otimes T$$

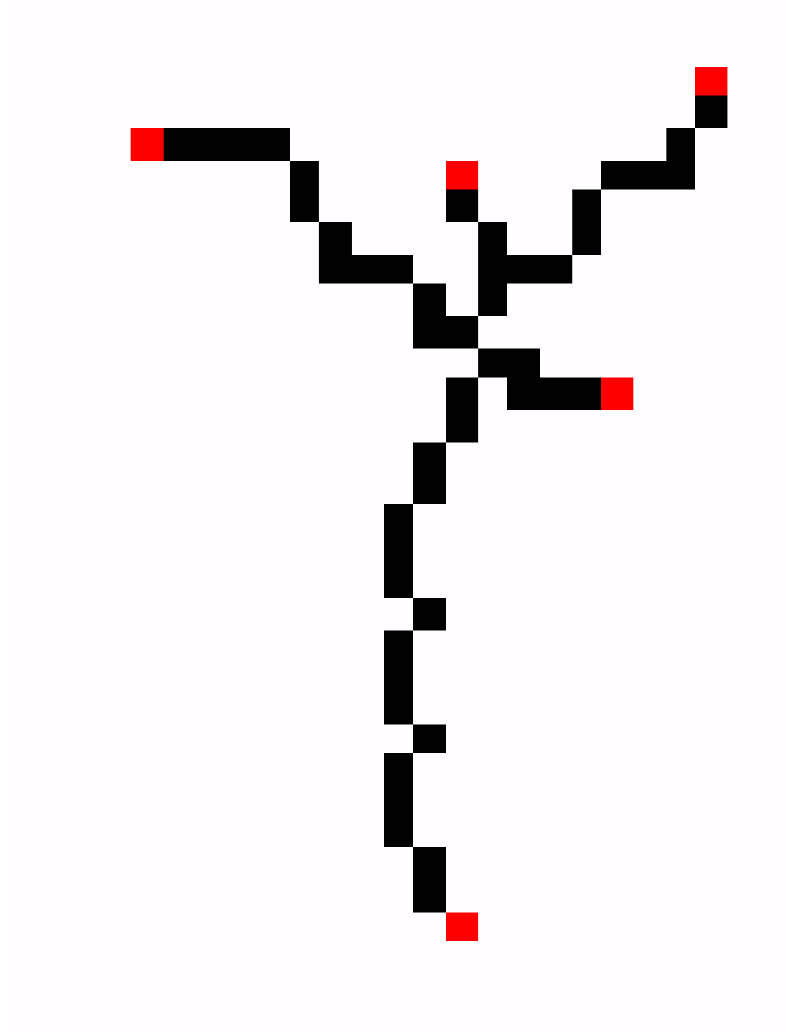
For $T' = (T_2, T_1)$:

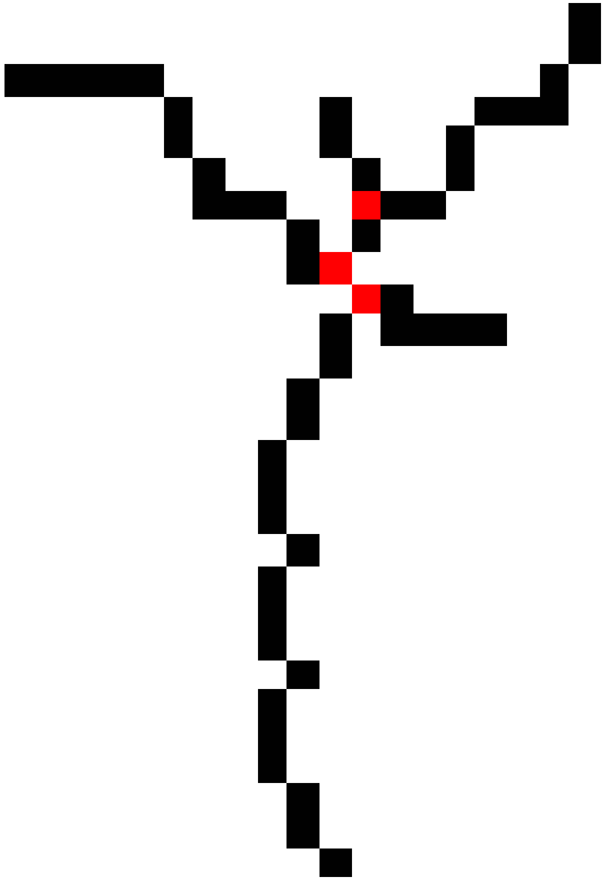
$$X \circ T = (X^C \odot T')^C$$

HMT: examples



HMT: examples





Skeleton: requirements

- compact representation of objects
- thin lines
- included and centered in the object
- homotopic to the object
- good representation of the geometry
- invertible (reconstruction of the initial object)

Skeleton: continuous case

A: open set
$s_\rho(A)$ = set of centers of maximal balls of A of radius ρ

Skeleton:

$$r(A) = \bigcup_{\rho > 0} s_\rho(A)$$

Characterization:

$$s_\rho(A) = \bigcap_{\mu > 0} [E(A, B_\rho) \setminus [E(A, B_\rho)]_{\bar{B}_\mu}]$$

$$r(A) = \bigcup_{\rho > 0} \bigcap_{\mu > 0} [E(A, B_\rho) \setminus [E(A, B_\rho)]_{\bar{B}_\mu}]$$

Reconstruction:

$$A = \bigcup_{\rho > 0} D(s_\rho, B_\rho)$$

Properties of the continuous skeleton

- $s_\rho(E_{\rho_0}(A)) = s_{\rho+\rho_0}(A) \;\Rightarrow\; r(E_{\rho_0}(A)) = \cup_{\rho>\rho_0} s_\rho(A)$
- no general formula for the skeleton of the dilation, opening or closing of a set
- $A \mapsto \bar{r}(A)$ is l.s.c. from $\mathcal{G}$ in $\mathcal{F}$
- A connected $\Rightarrow\; \bar{r}(A)$ connected
- the skeleton is “thin”: its interior is empty

Skeleton: digital case

- Direct transposition of the continuous case:

$$S(X) = \bigcup_{n \in \mathbb{N}} [E(X, B_n) \setminus E(X, B_n)_B]$$

Properties:

- centers of digital maximal balls
- reconstuction
- but poor connectivity properties

Skeleton: digital case

- Direct transposition of the continuous case:

$$S(X) = \bigcup_{n \in \mathbb{N}} [E(X, B_n) \setminus E(X, B_n)_B]$$

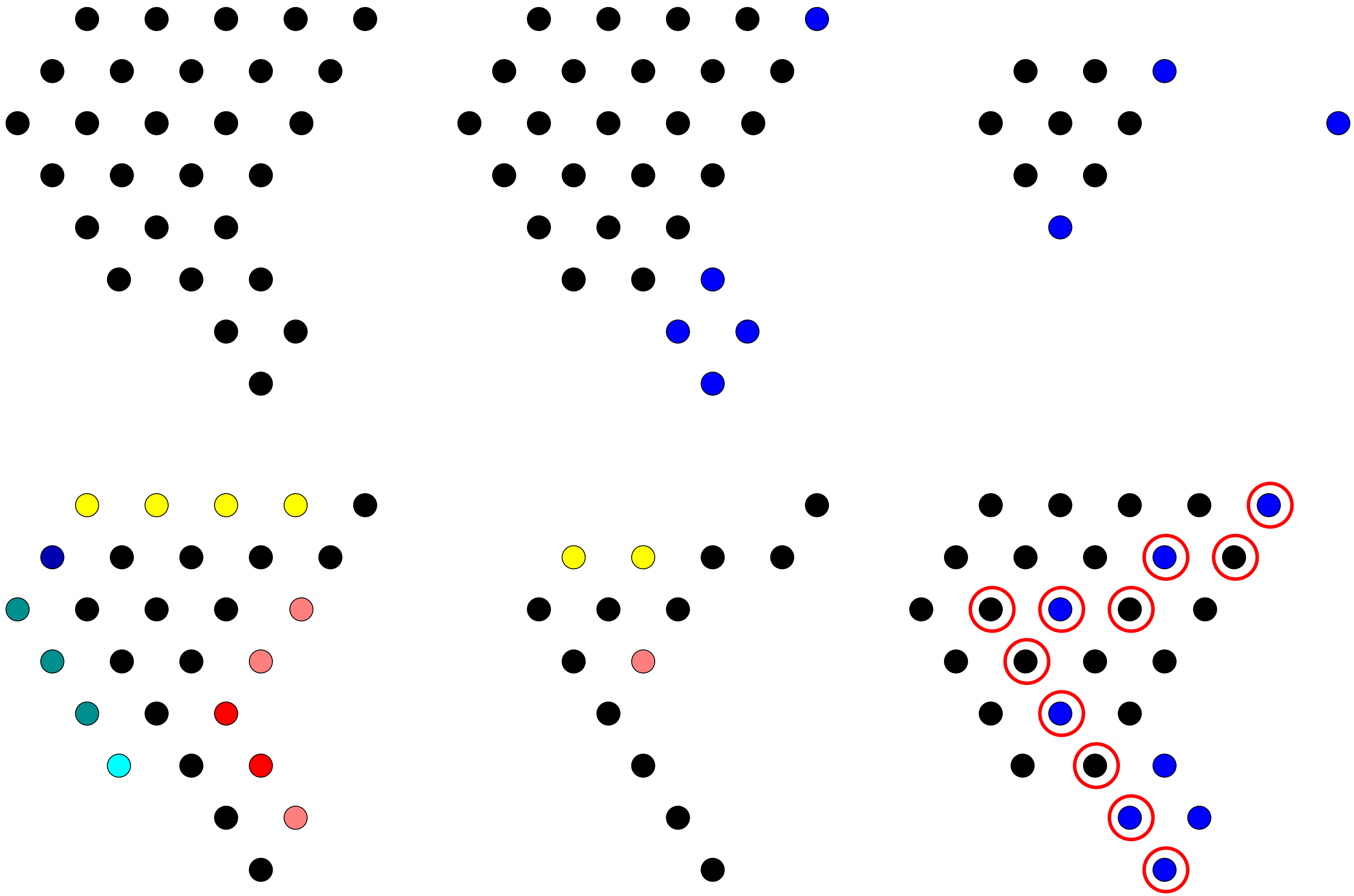
  Properties:
  - centers of digital maximal balls
  - reconstuction
  - but poor connectivity properties

- Skeleton from homotopic thinning

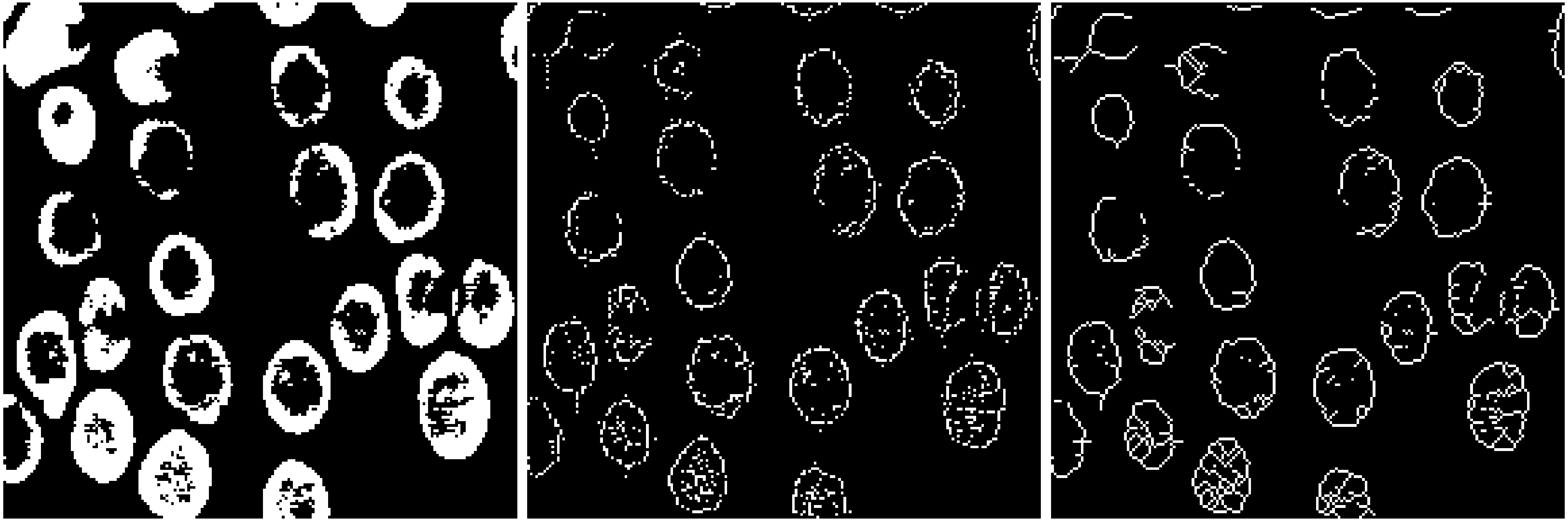
$$\begin{array}{ccccc} & 1 & & 1 & \\ . & & 1 & & . \\ & 0 & & 0 & \end{array}$$

  Properties:
  - perfect topology
  - no reconstruction

Centers of maximal ball vs thinning



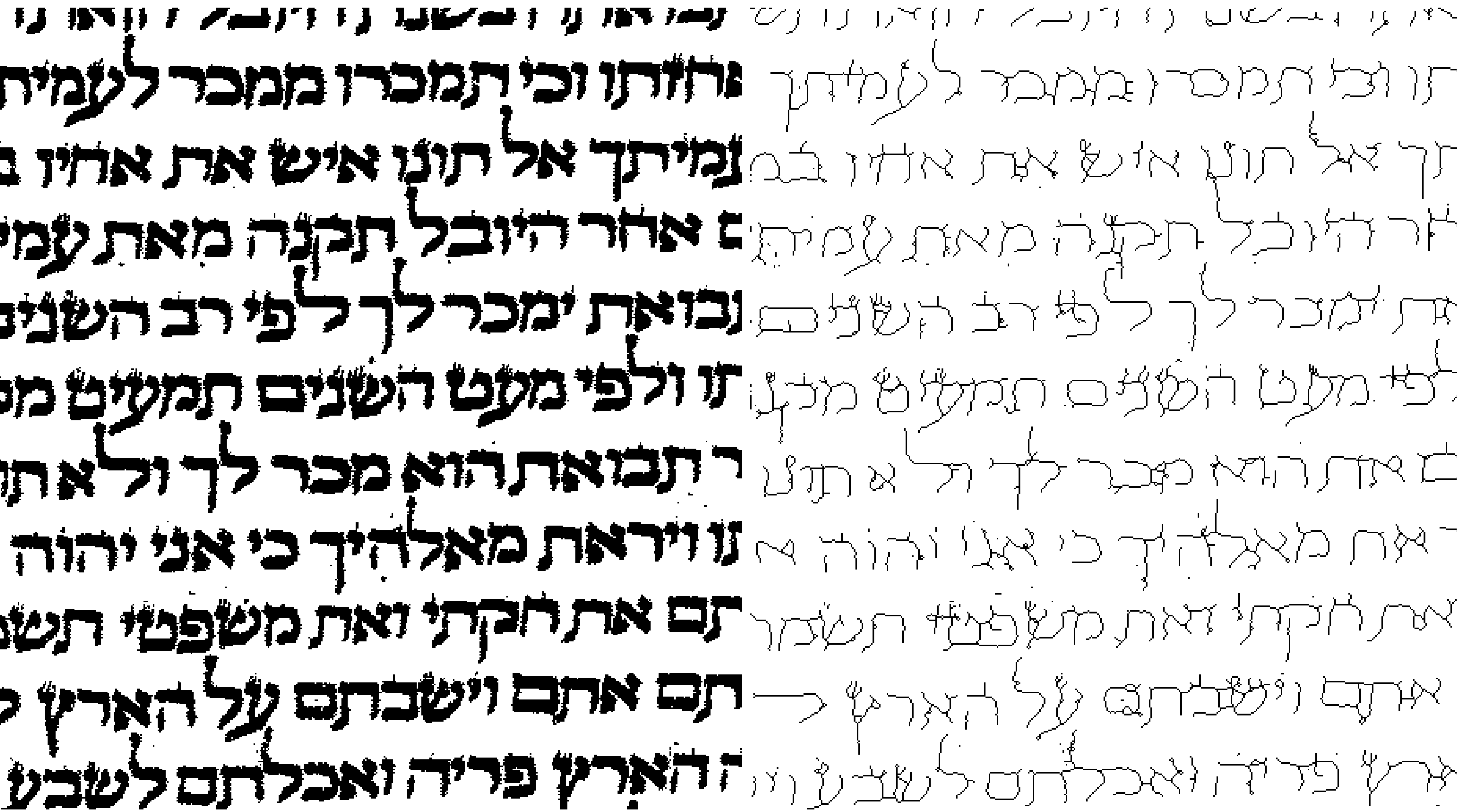
Centers of maximal ball vs thinning



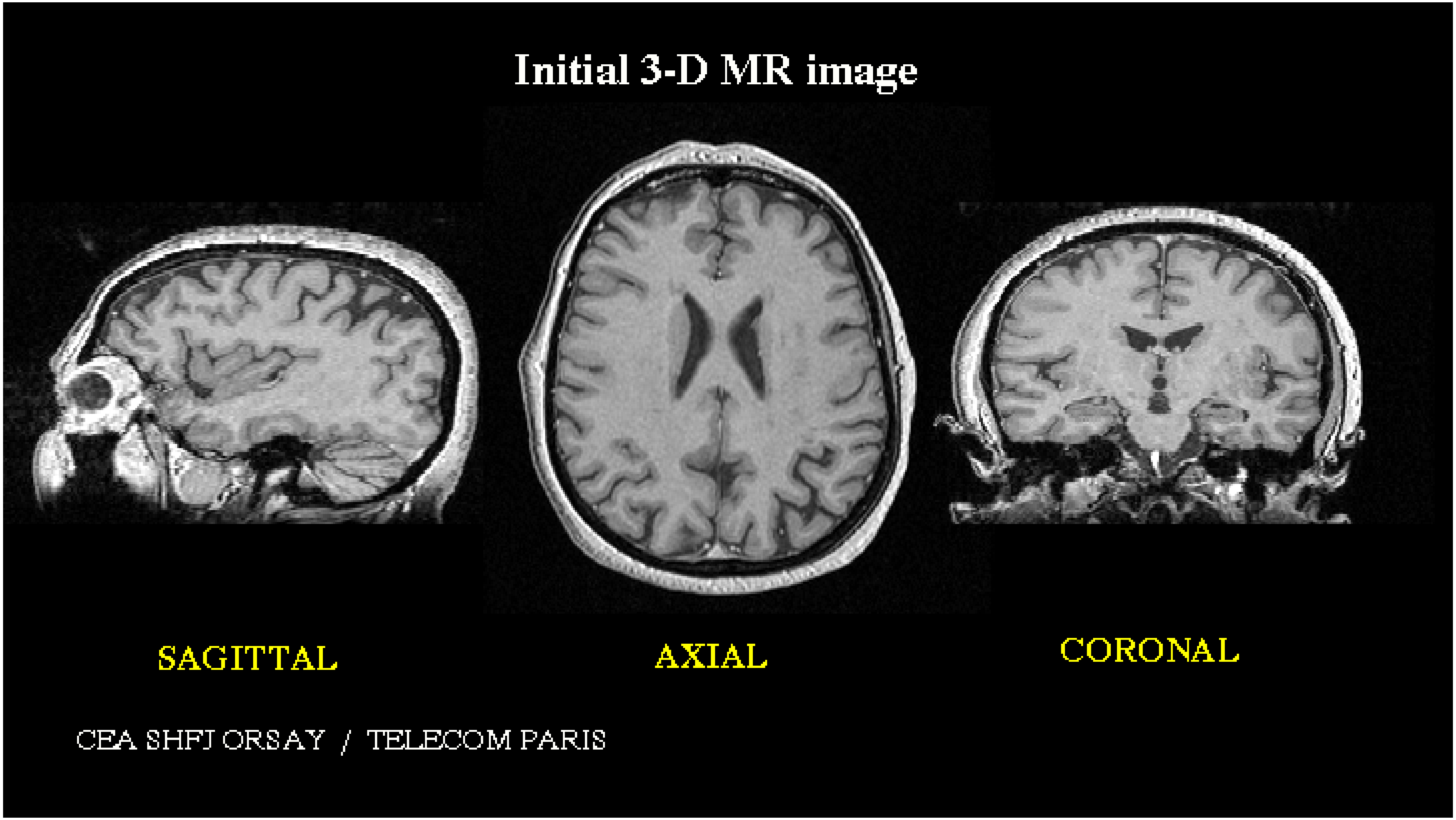
Centers of maximal ball vs thinning

Centers of maximal ball vs thinning

Centers of maximal ball vs thinning



Application in brain imaging (PhD of Jean-François Mangin)



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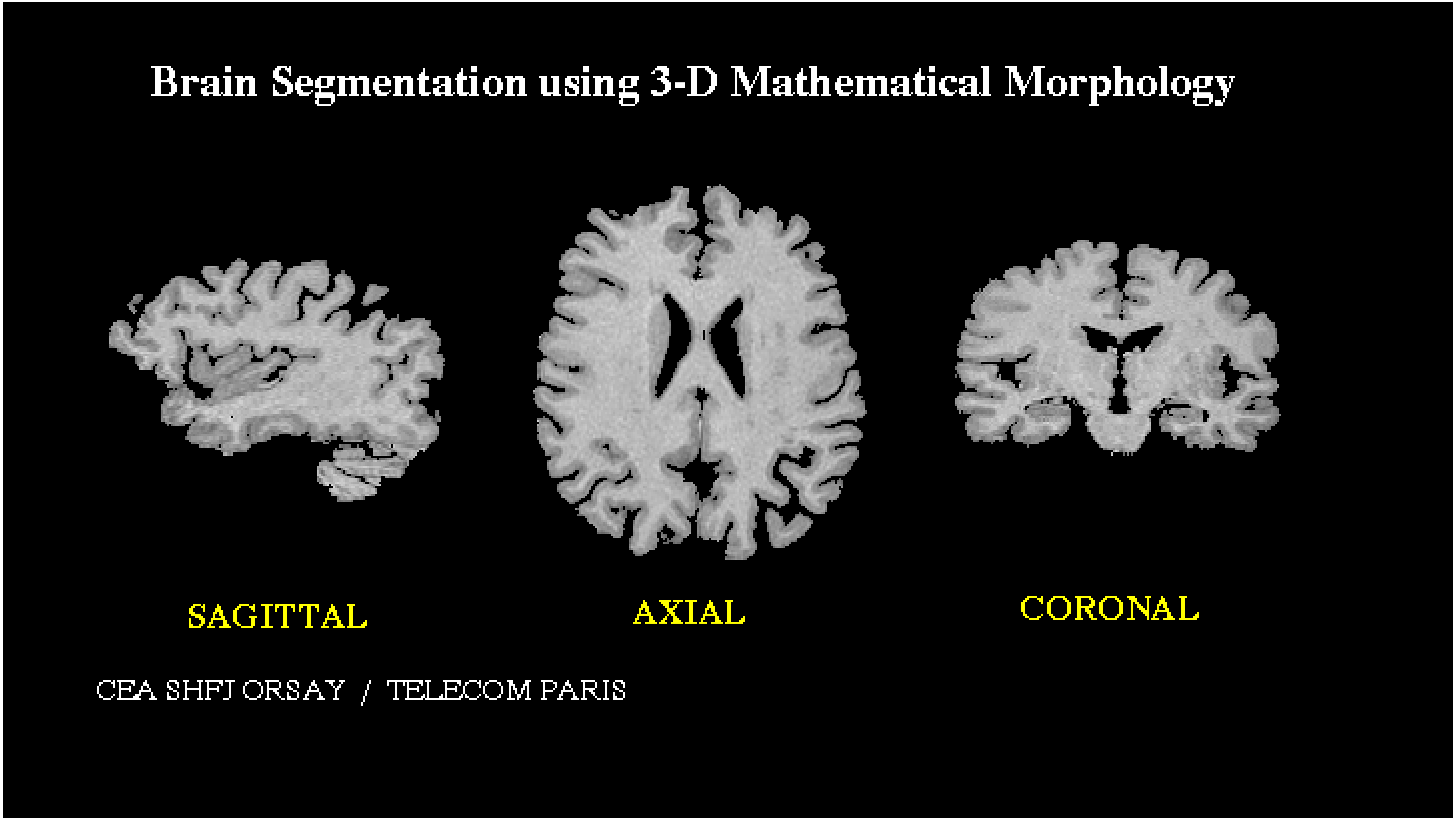

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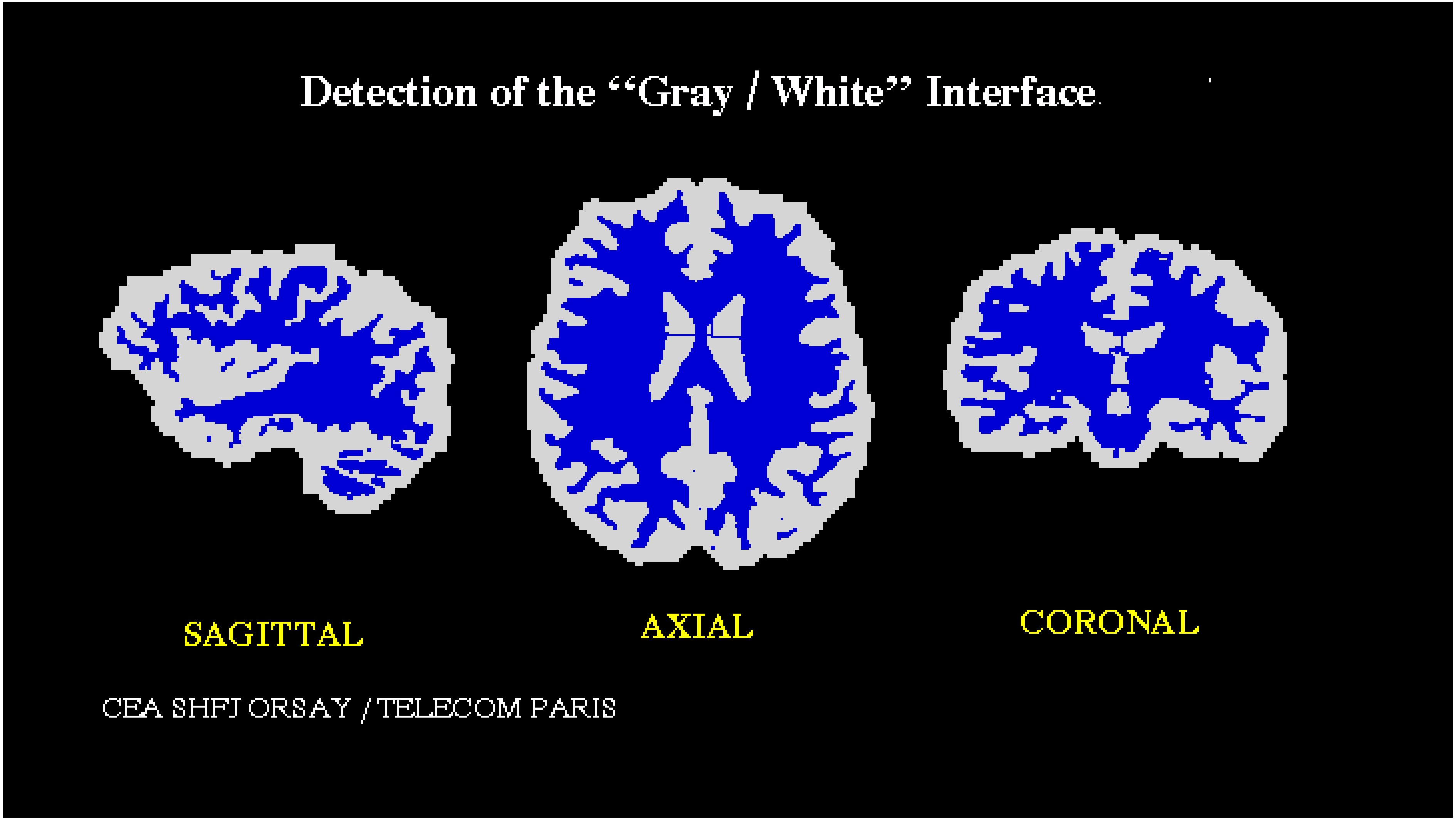

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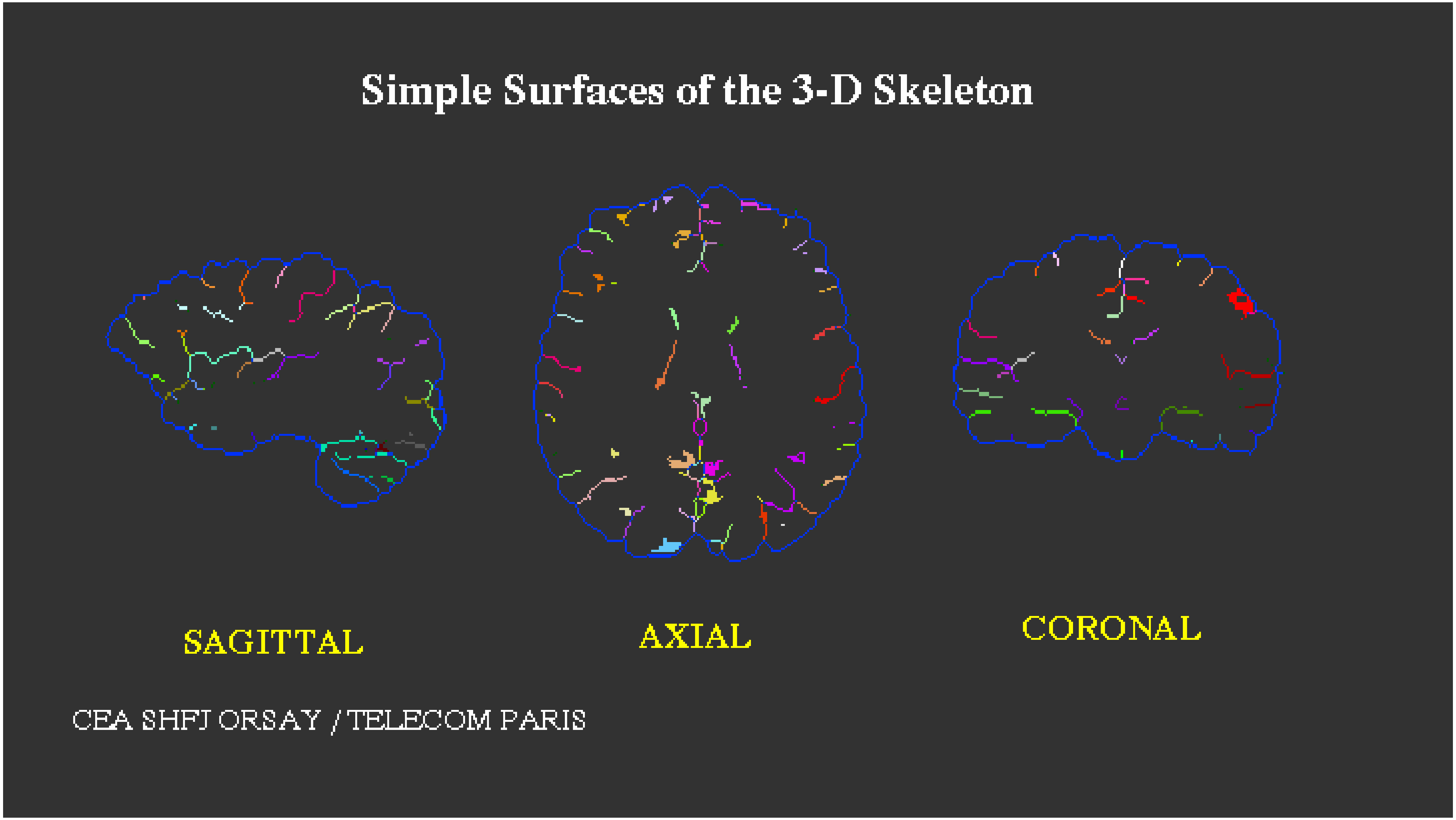

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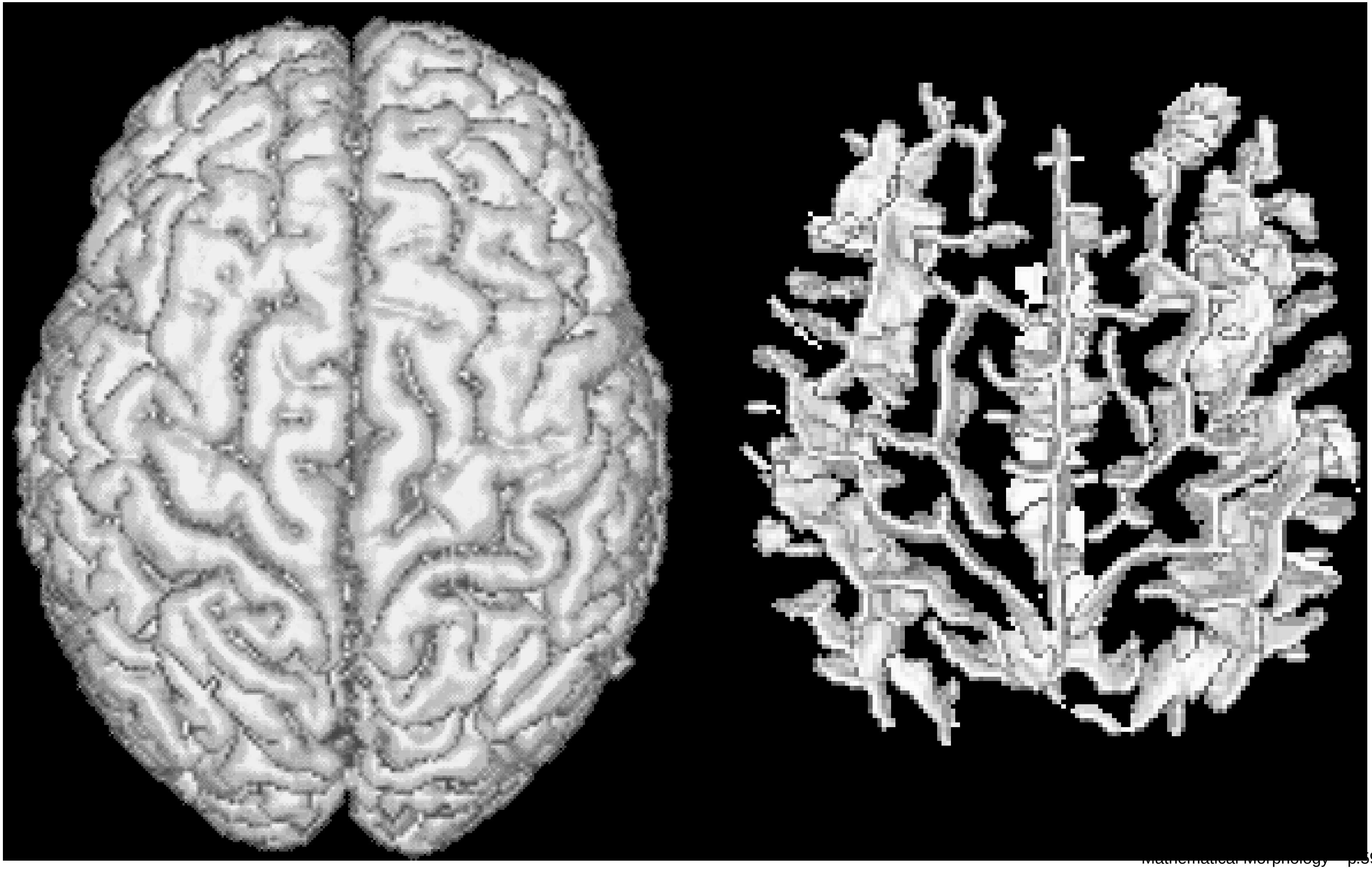
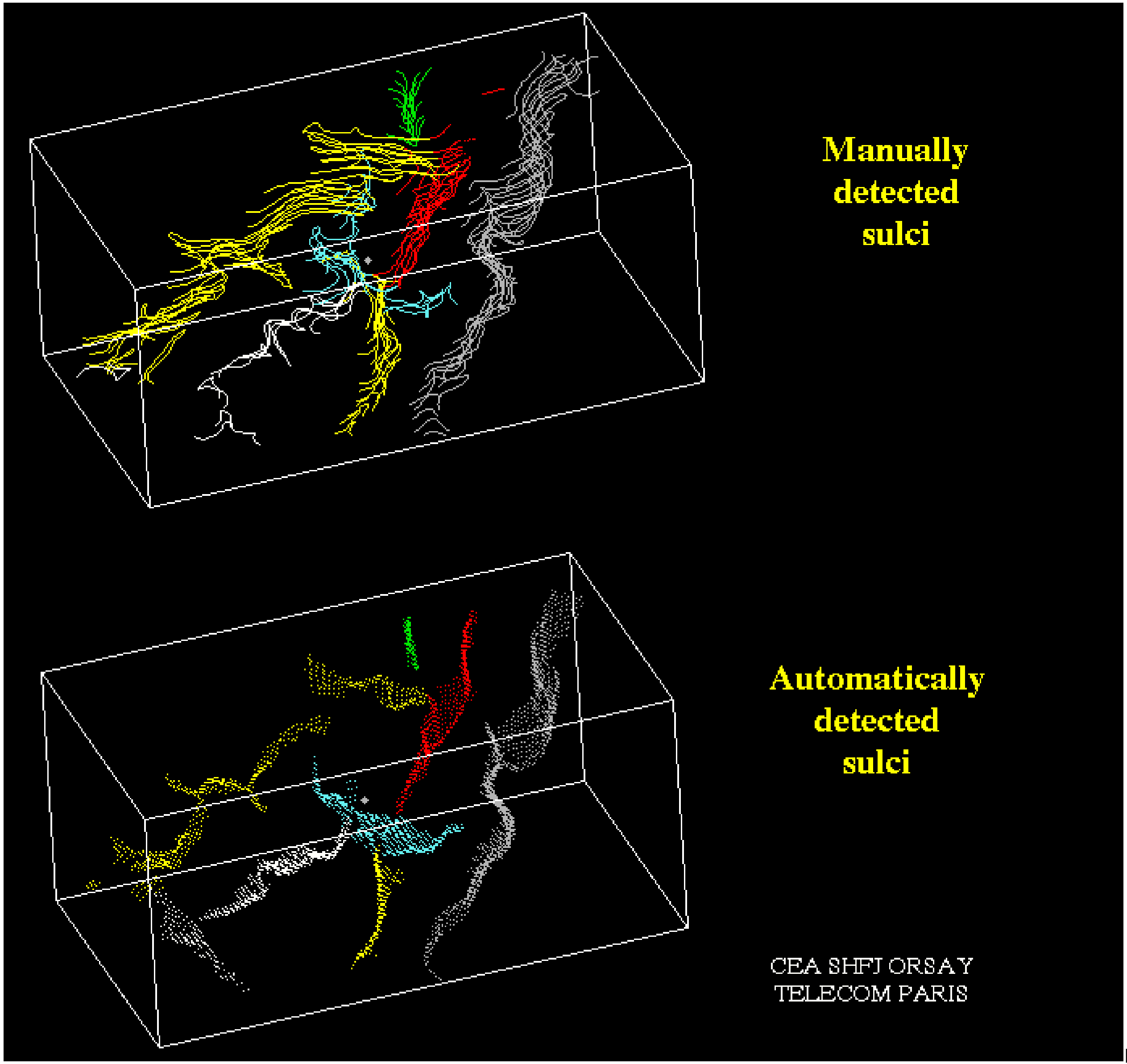


Application in brain imaging (PhD of Jean-François Mangin)



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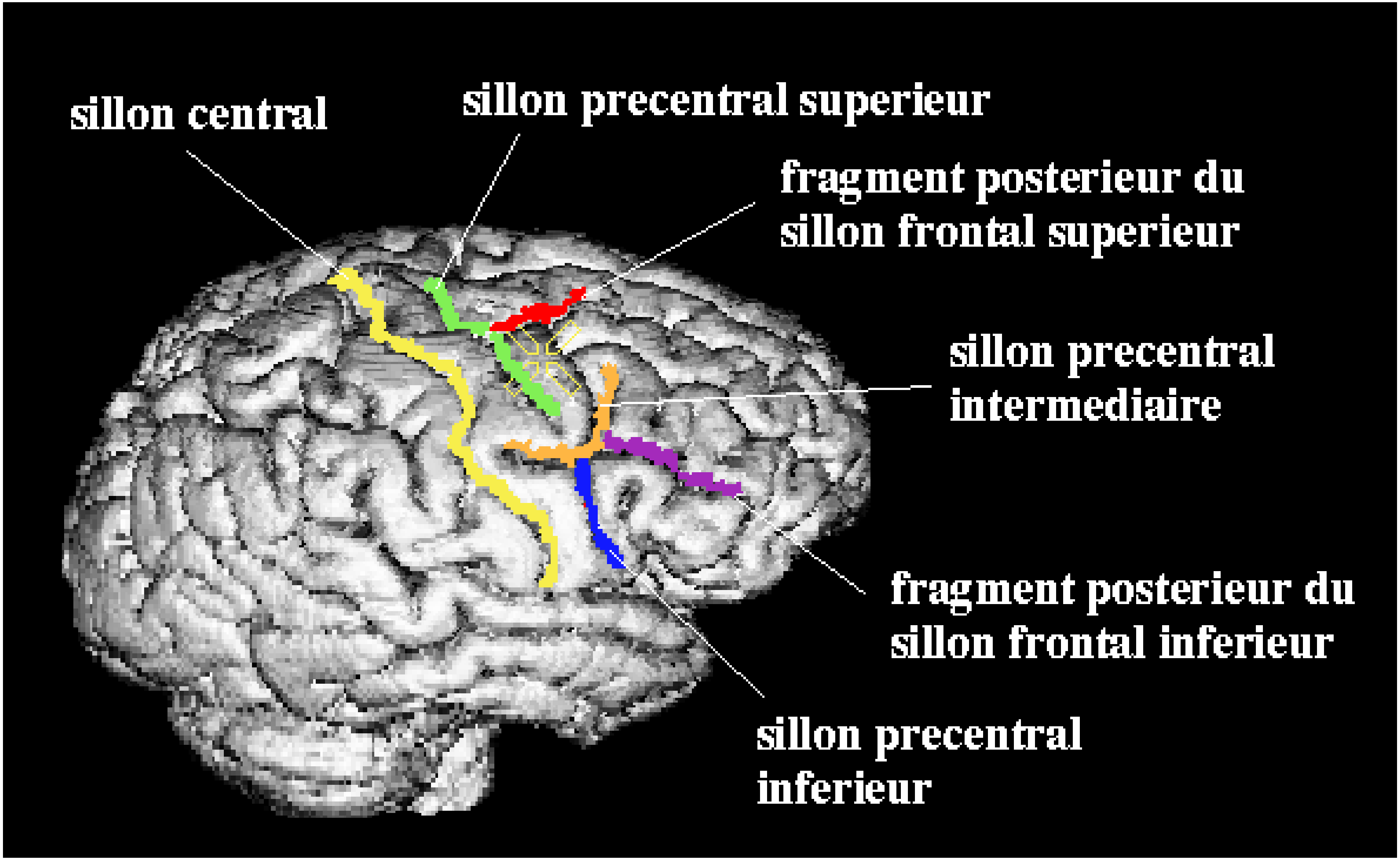
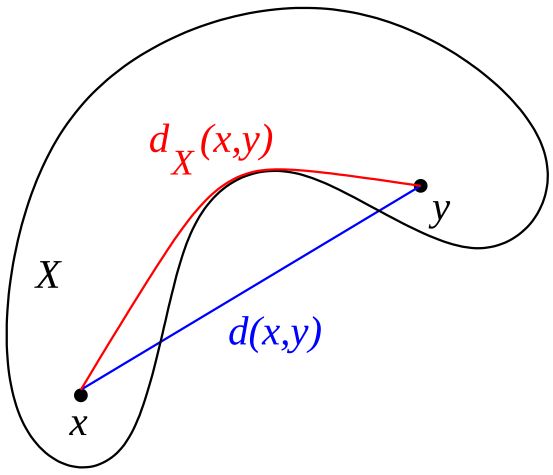


Geodesic operators



Geodesic distance, conditional to X: d_X

- if X is closed, there exits a geodesic arc for any pair of points of X
- unique if X is simply connected
- X convex $\Leftrightarrow$ $d_X = d$

Geodesic ball: $B_X(x,r) = \{y \in X \ / \ d_X(x,y) \leq r\}$
Rq: $B_X(x,r) \subseteq B(x,r)$

Geodesic dilation:

$$D_X(Y, B_r) = \{x \in \mathbb{R}^n \ / \ B_X(x,r) \cap Y \neq \emptyset\} = \{x \in \mathbb{R}^n \ / \ d_X(x,Y) \leq r\}$$

Geodesic erosion:

$$E_X(Y, B_r) = \{x \in \mathbb{R}^n \ / \ B_X(x,r) \subseteq Y\} = X \setminus D_X(X \setminus Y, B_r)$$

Geodesic opening and closing: by composition

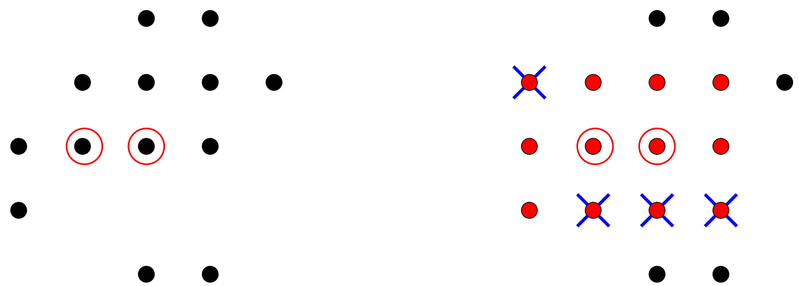
Properties and reconstruction

Properties:

- similar as in the Euclidean case
- $D_X(Y, B_r) \subseteq D(Y, B_r)$
- $D_X(Y, B_r) = \cap_{n=1}^{\infty} [(Y \oplus \frac{r}{n} B) \cap X]^n$

Digital case:

$$D_X(Y, B_r) = [D(Y, B_1) \cap X]^r$$

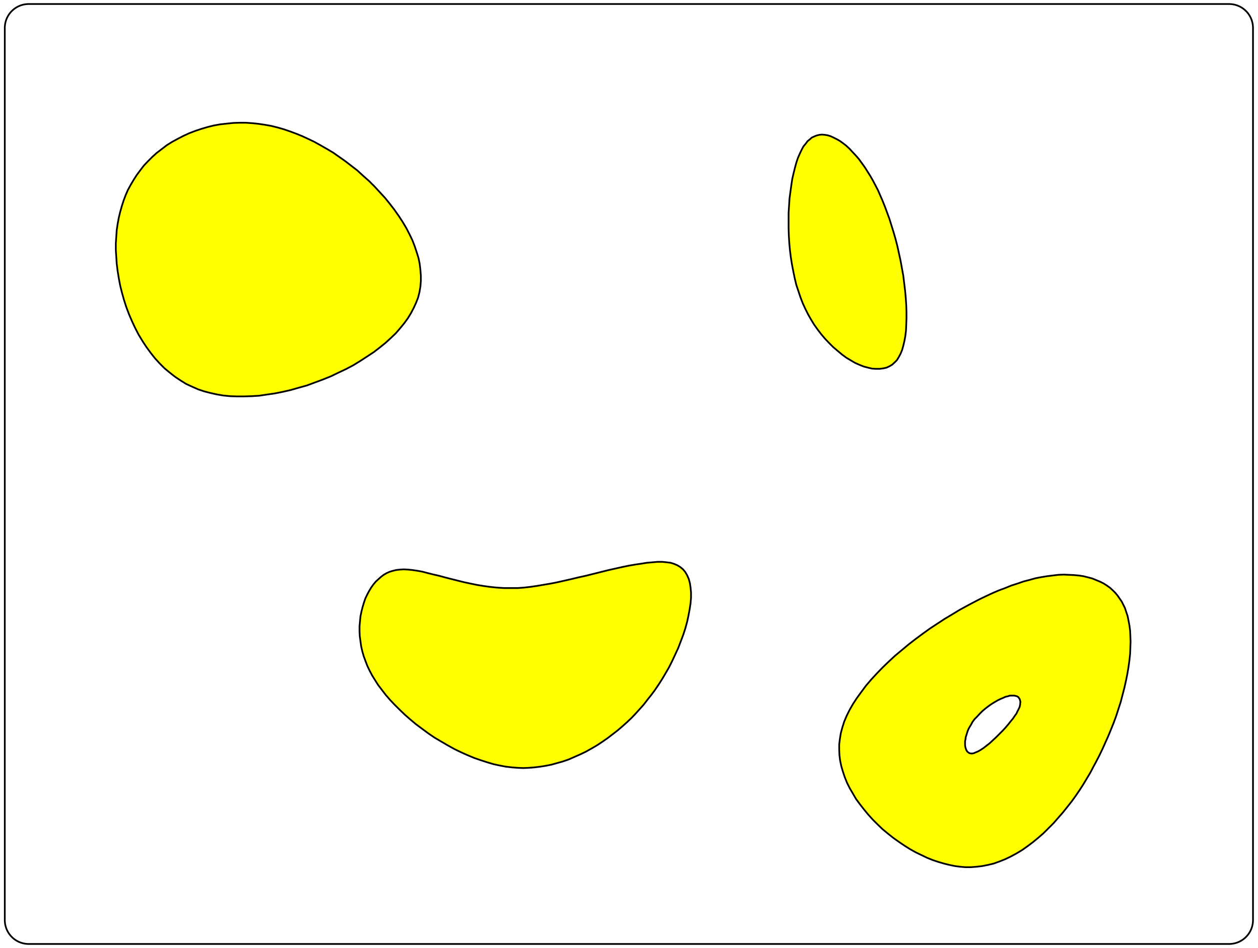


Reconstruction:

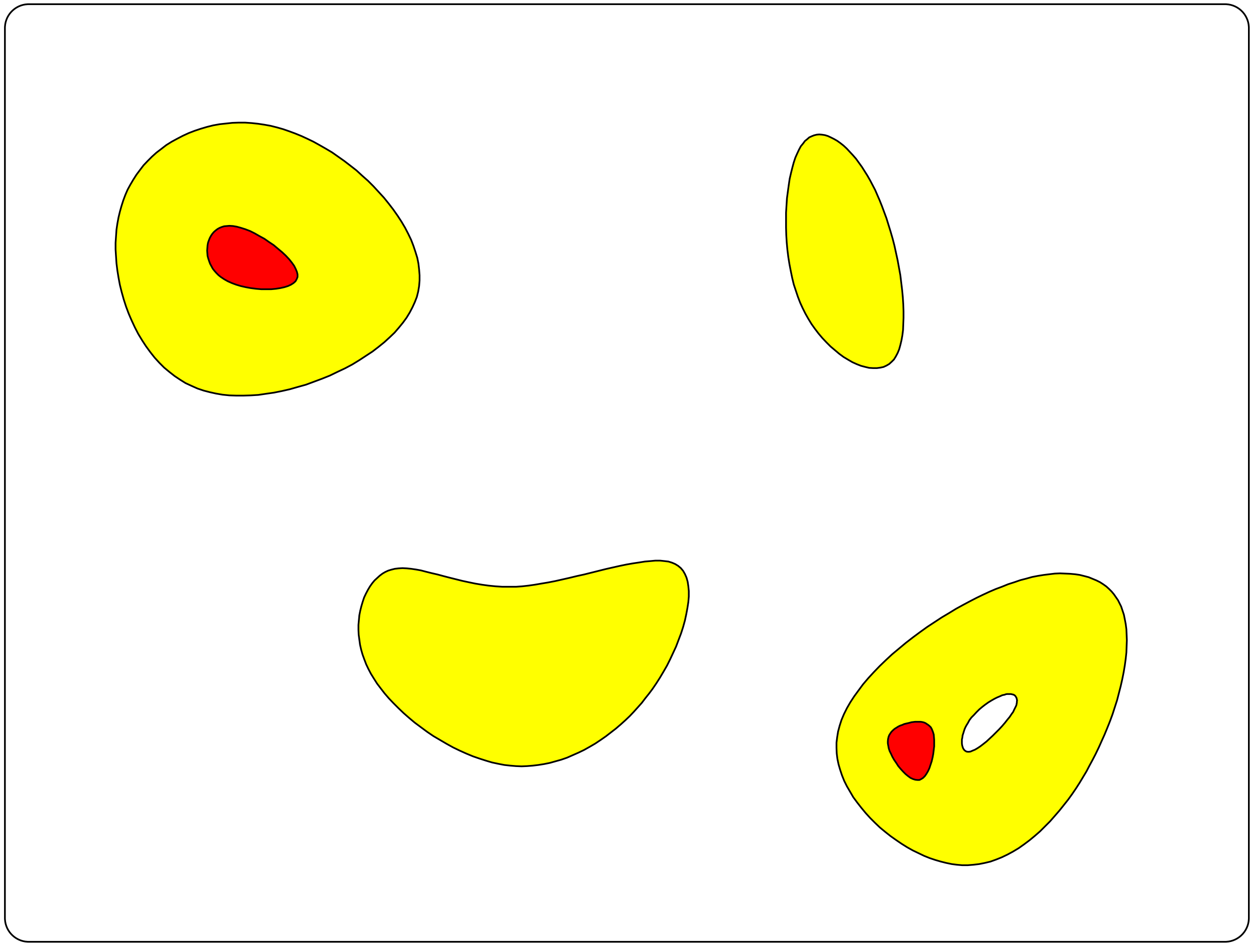
$$[D(Y, B_1) \cap X]^{\infty} = D_X^{\infty}(Y)$$

= connected components of X which intersect Y

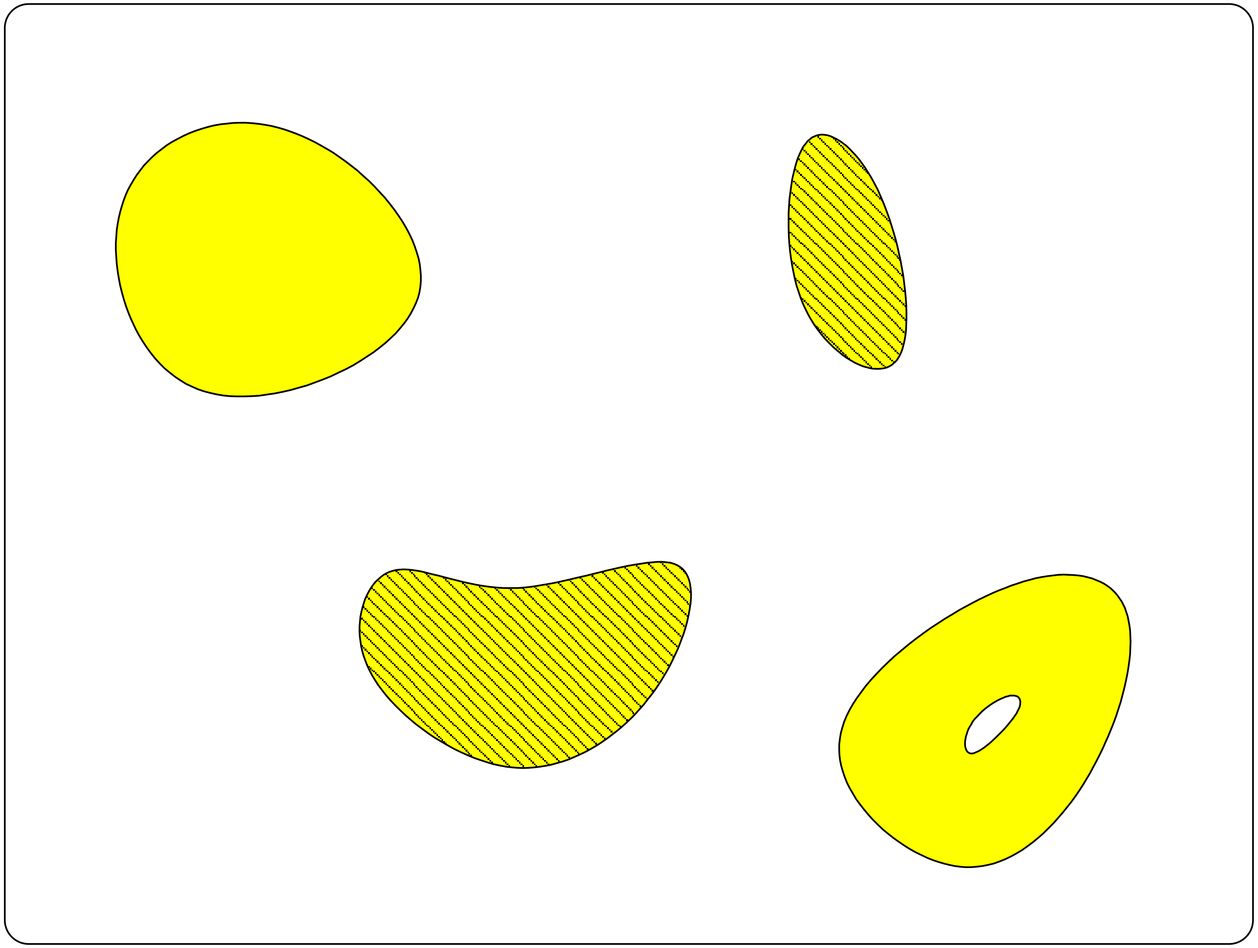
Binary reconstruction: example



Binary reconstruction: example



Binary reconstruction: example



Geodesic operators on functions

$$X_1 \subseteq X_2 \text{ and } Y_1 \subseteq Y_2 \;\Rightarrow\; D_{X_1}(Y_1, B_r) \subseteq D_{X_2}(Y_1, B_r) \subseteq D_{X_2}(Y_2, B_r)$$

$\Rightarrow$ Extension to functions, for $f \leq g$, cut by cut:

$$[D_g(f, B_r)]_\lambda = D_{g_\lambda}(f_\lambda, B_r)$$

(with $f_\lambda = \{x,\ f(x) \geq \lambda\}$)

Digital case:

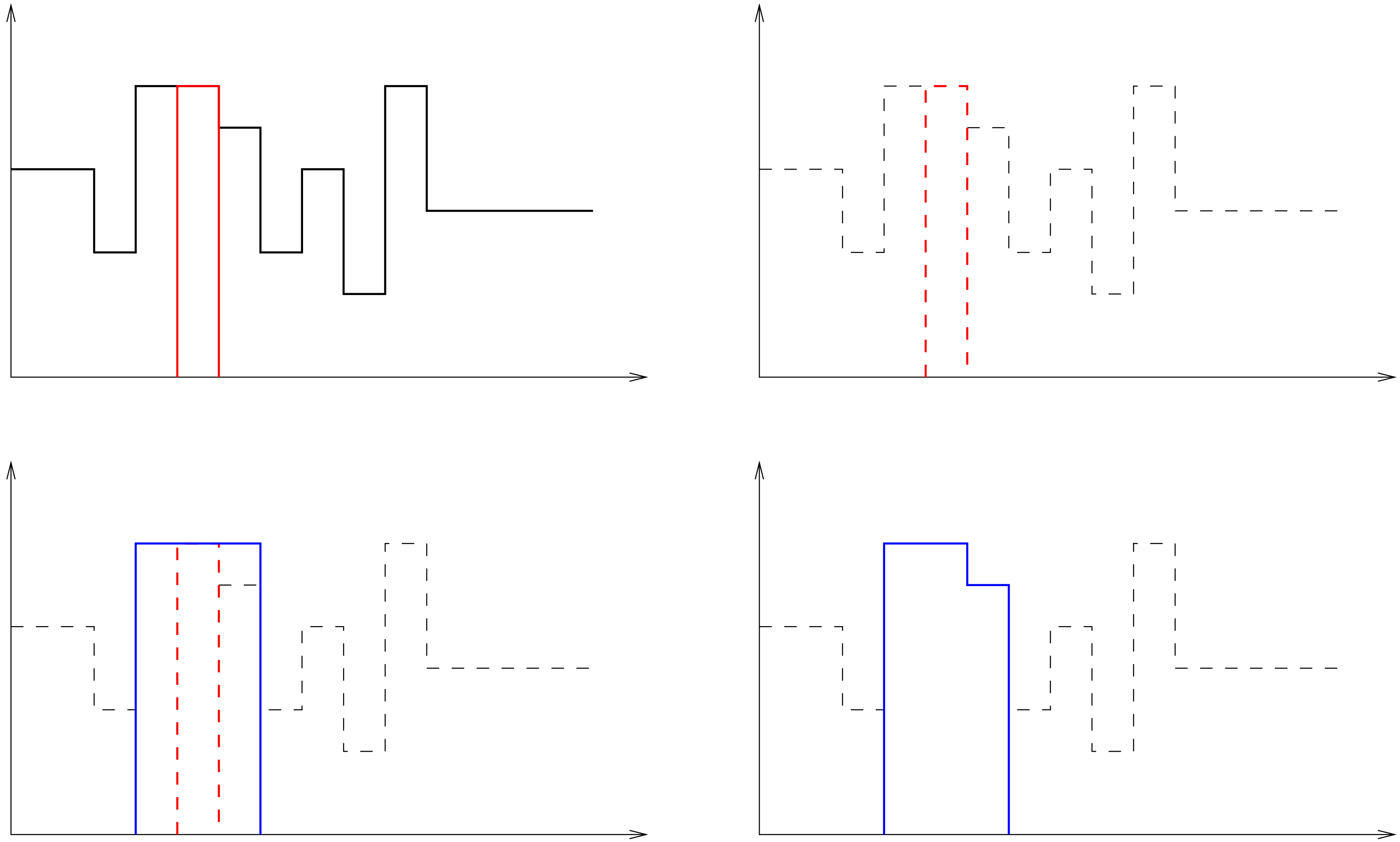
$$D_g(f, B_r) = [D(f, B_1) \wedge g]^r$$

$$E_g(f, B_r) = [E(f, B_1) \vee g]^r$$

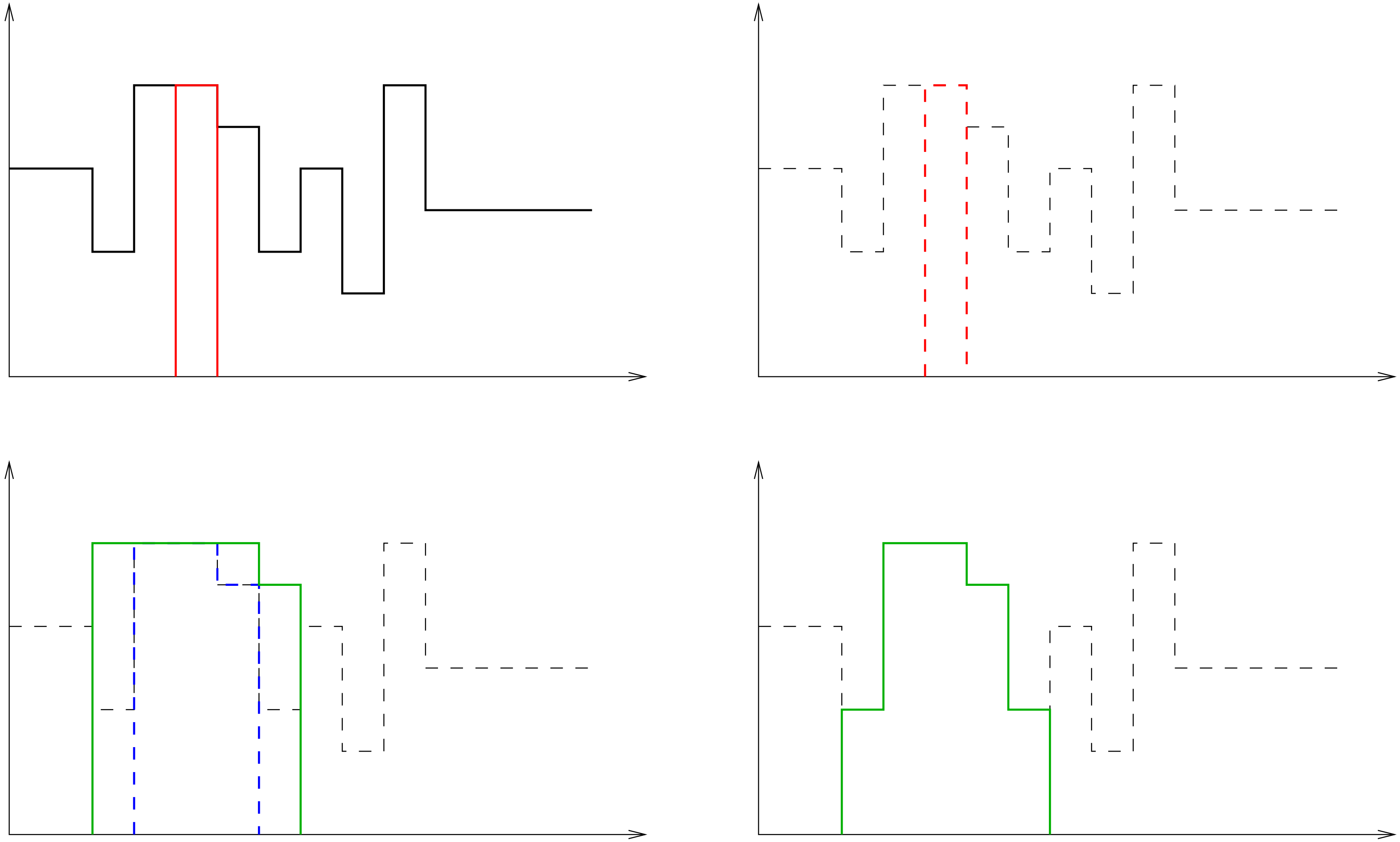
Numerical reconstruction of f (marker function) in g:

- by dilation $D_g(f, B_\infty) = D_g^\infty(f)$: opening
- by erosion $E_g(f, B_\infty)$: closing
- opening by reconstruction: $D_f^\infty(f_B)$ (flat areas whose contours are some contours of the original image $\Rightarrow$ compression)

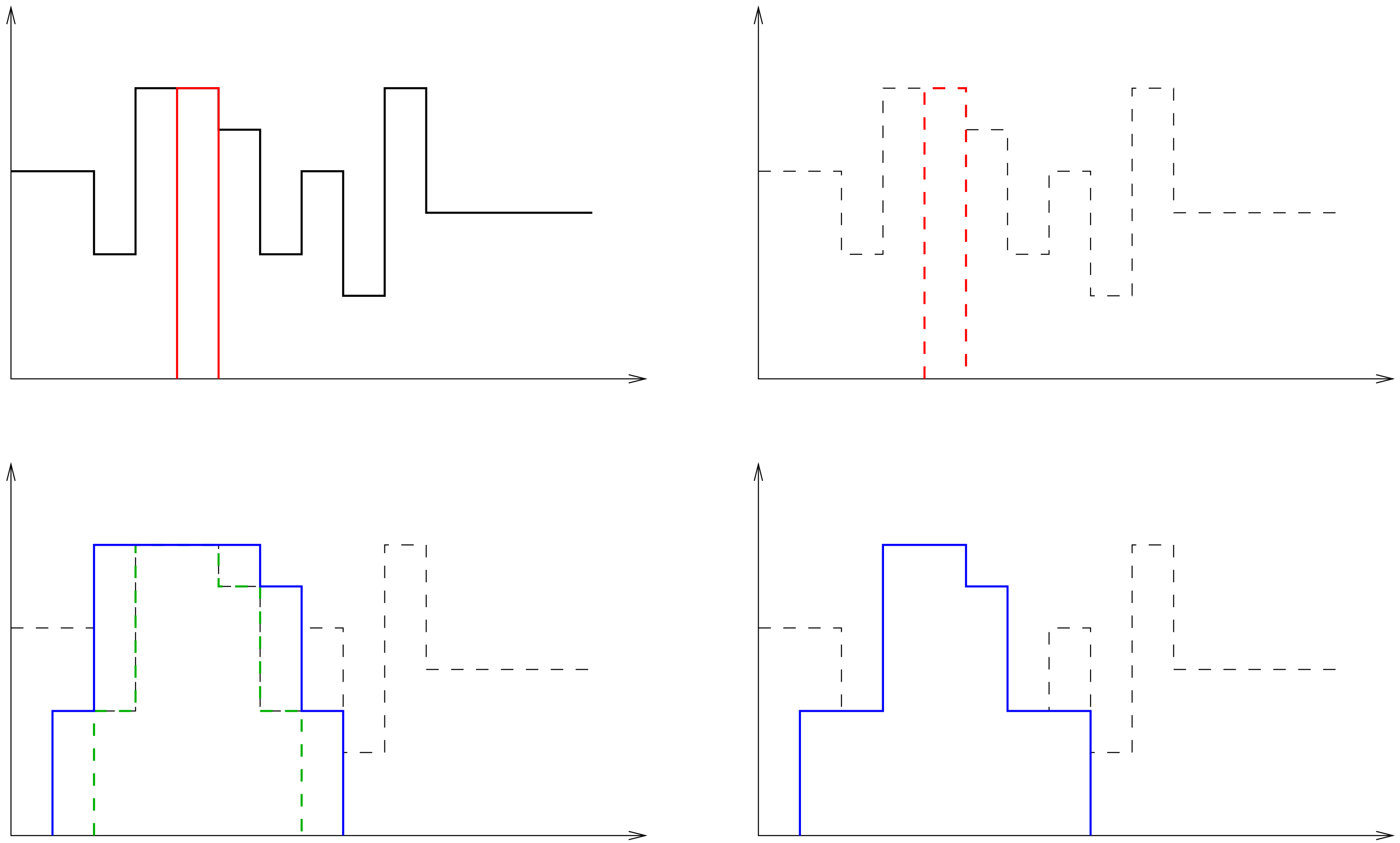
Numerical reconstruction: example



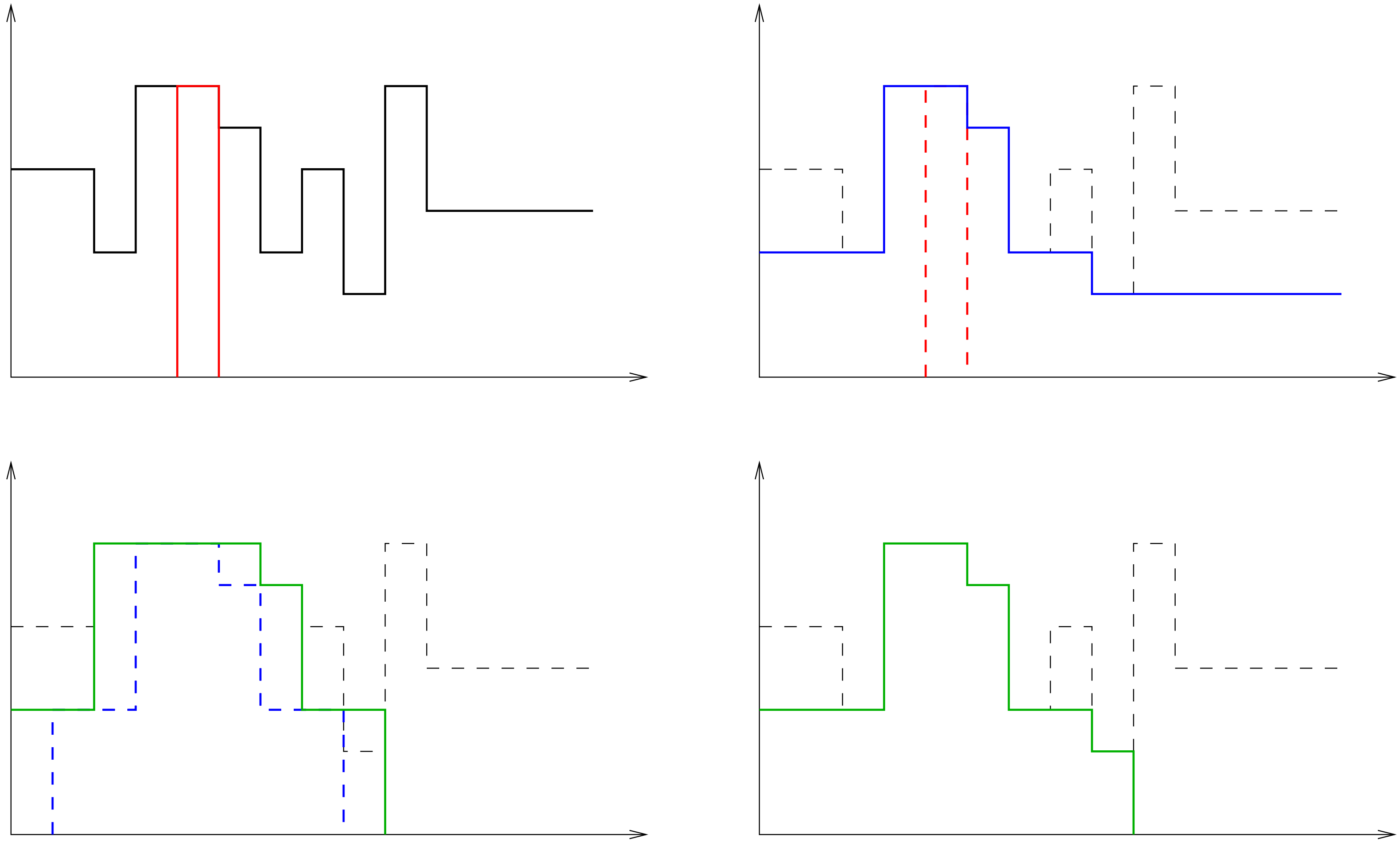
Numerical reconstruction: example



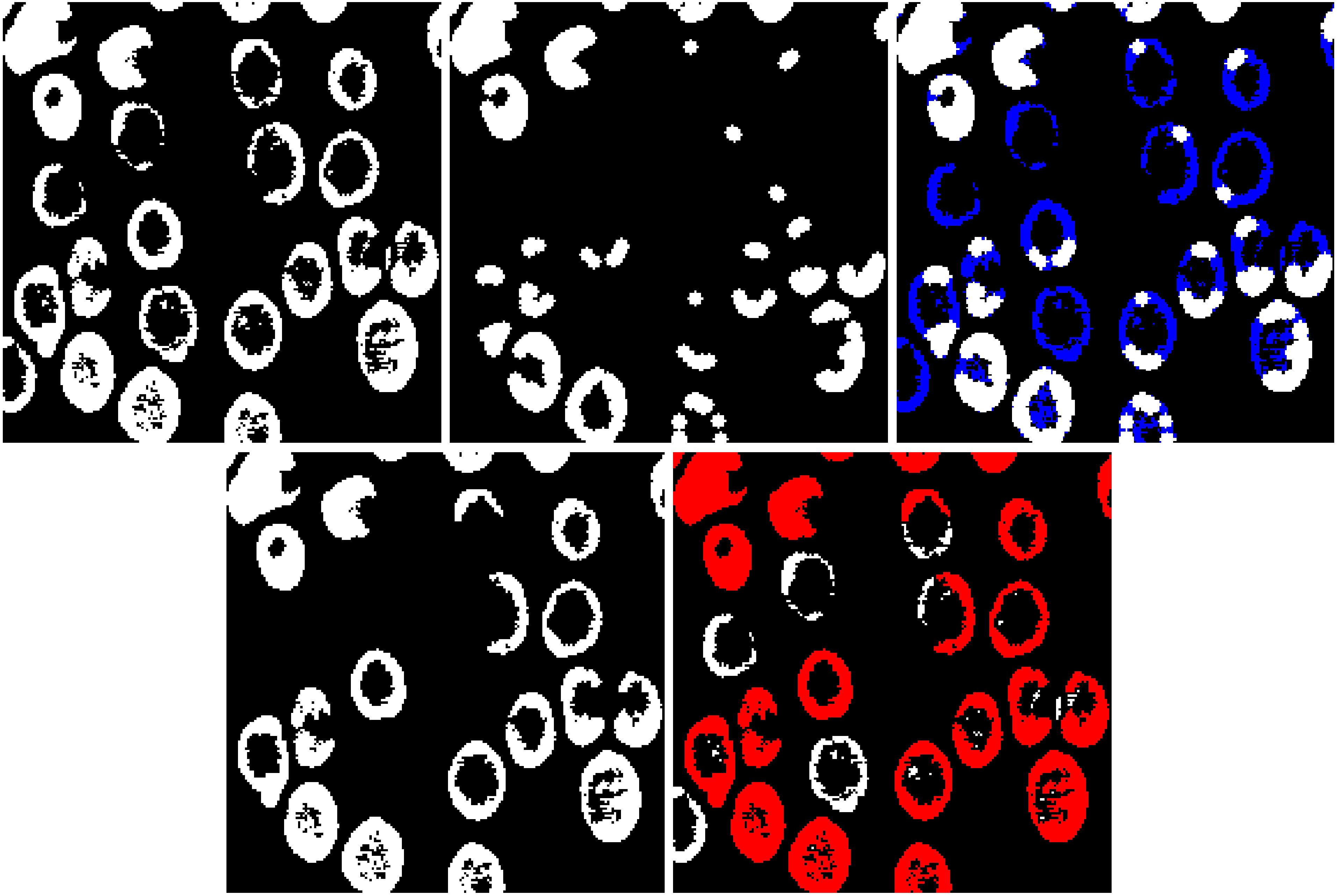
Numerical reconstruction: example



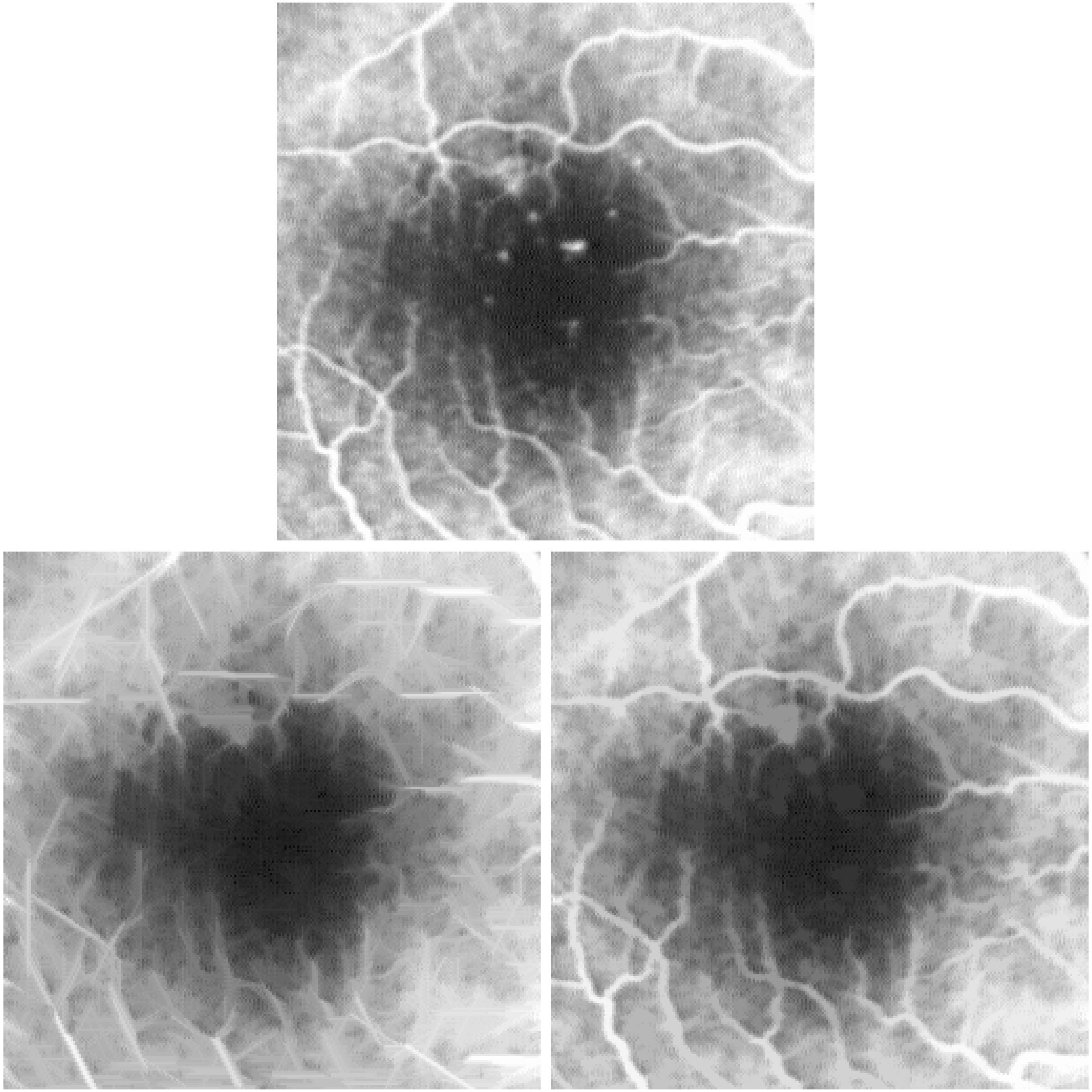
Numerical reconstruction: example



Opening by reconstruction: examples

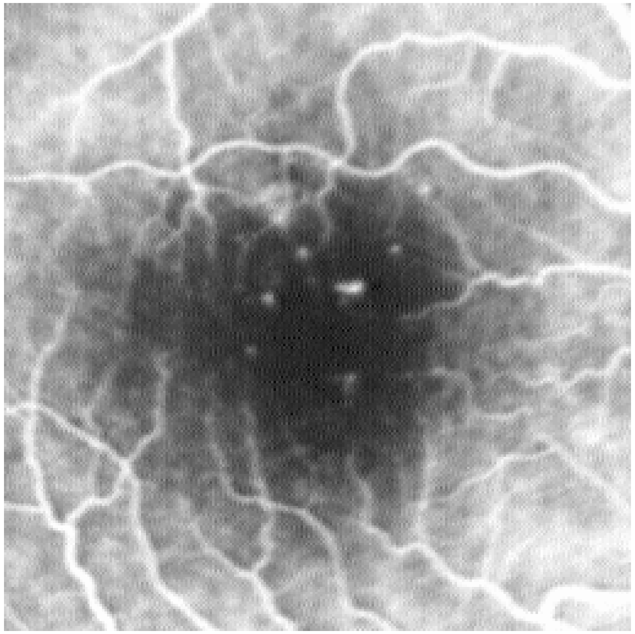


Opening by reconstruction: examples

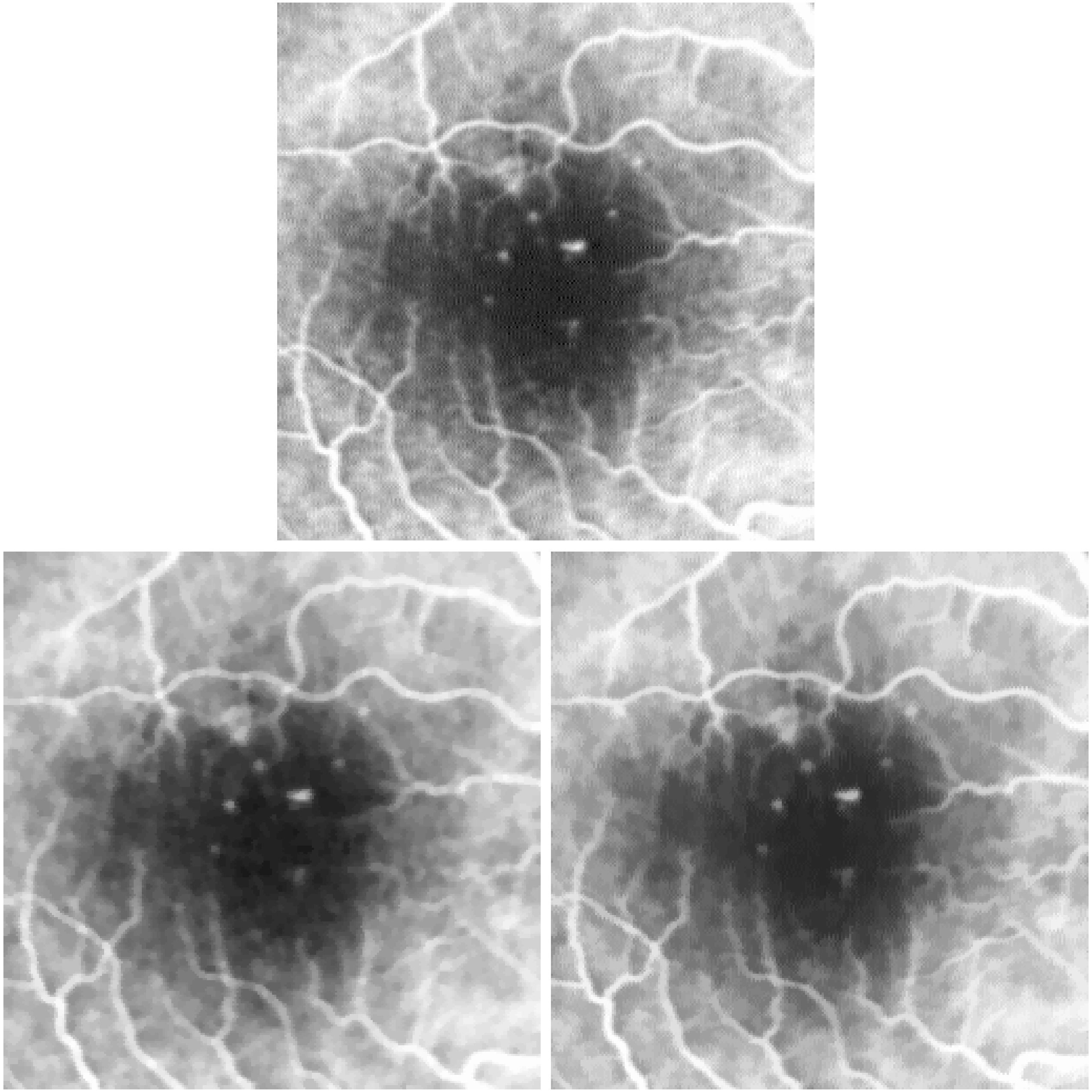


Union of openings by segments of length 20 and reconstruction

Application to alternate sequential filters

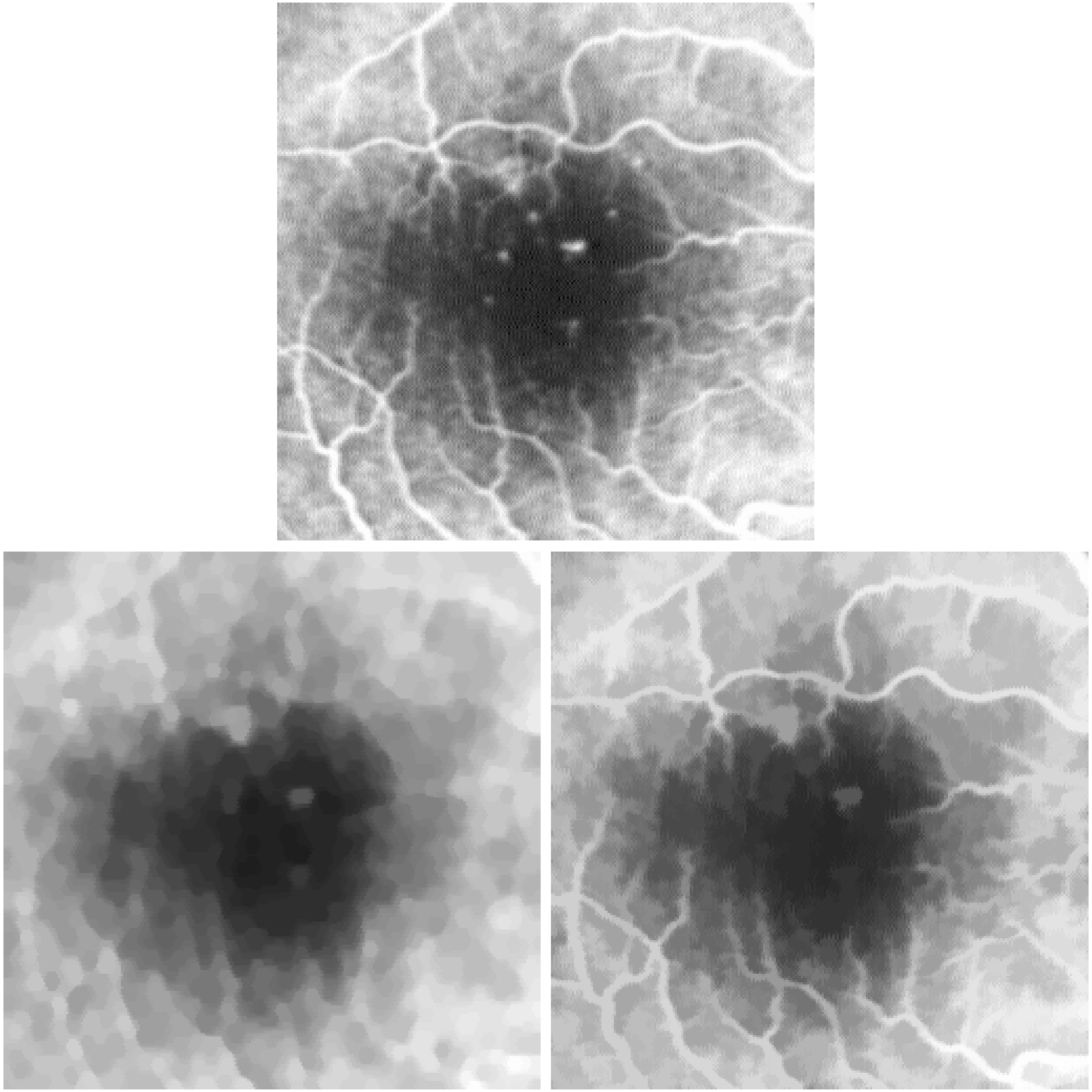


Application to alternate sequential filters



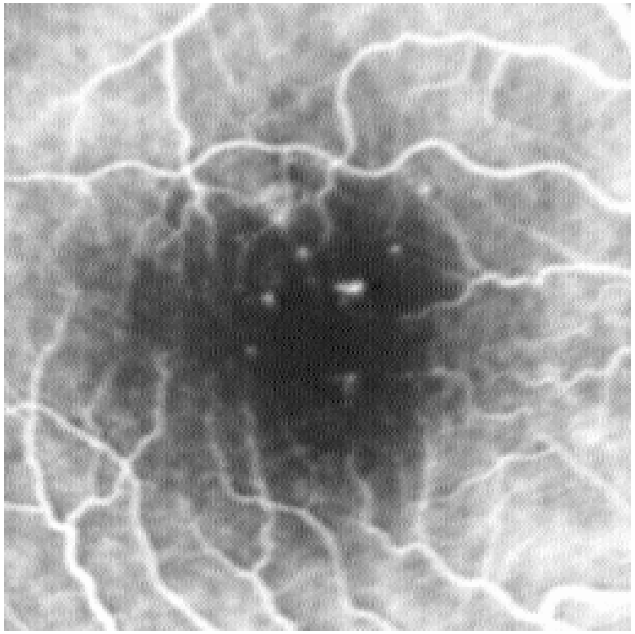
ASF with an hexagon (maximal size = 1)

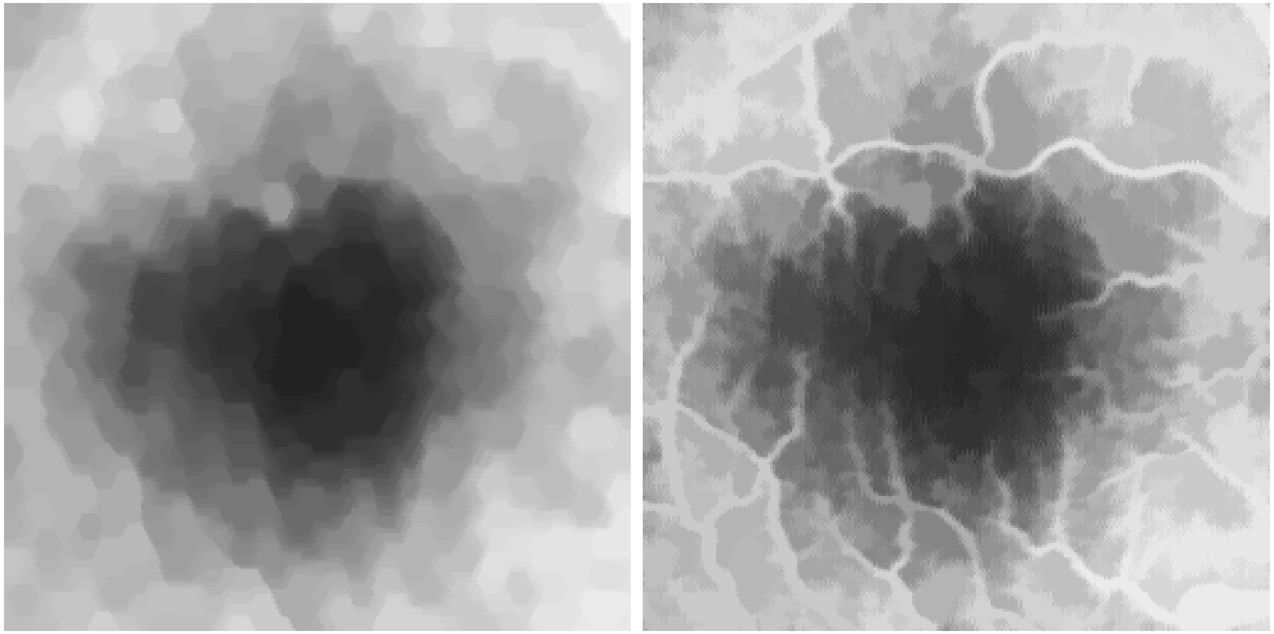
Application to alternate sequential filters



ASF with an hexagon (maximal size = 3)

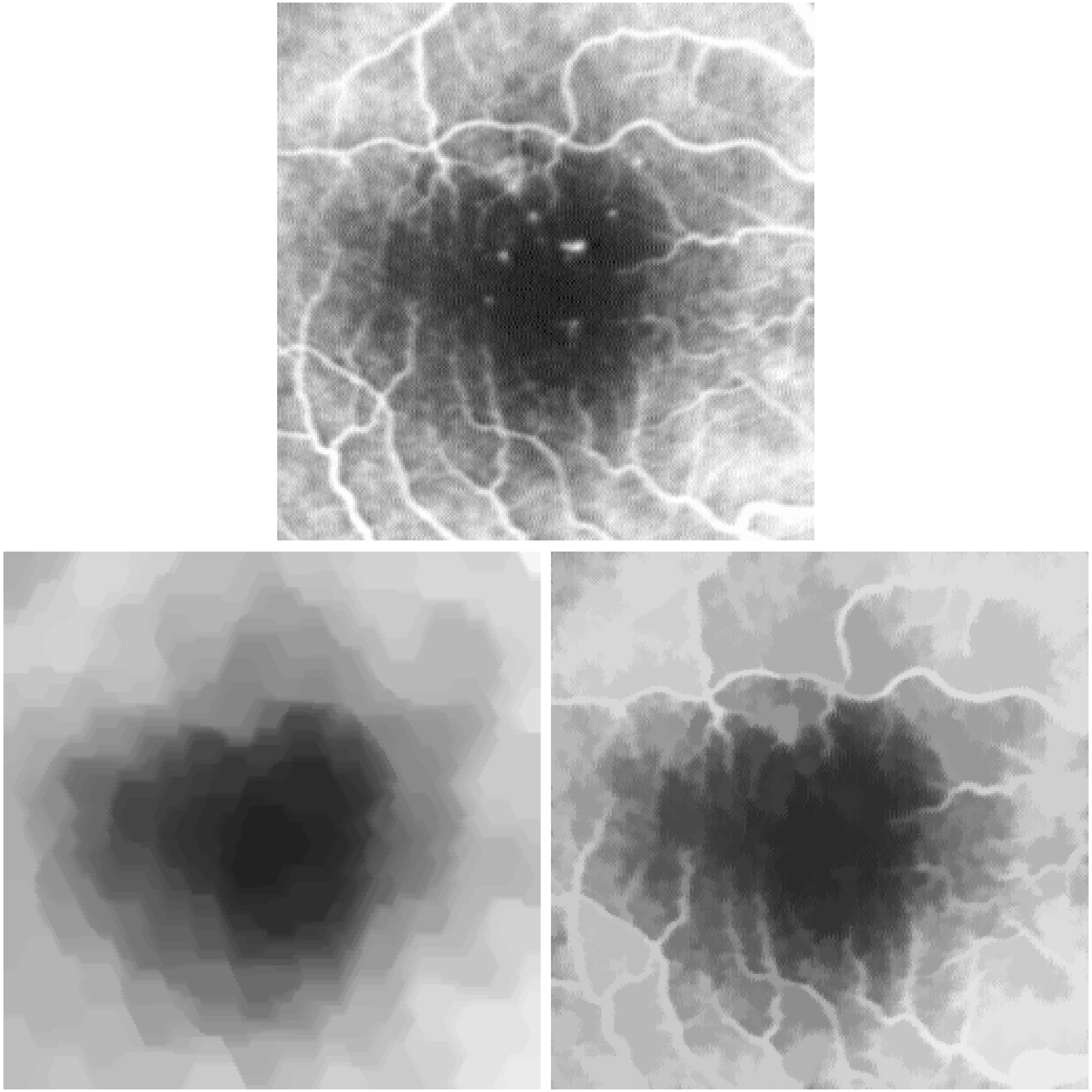
Application to alternate sequential filters





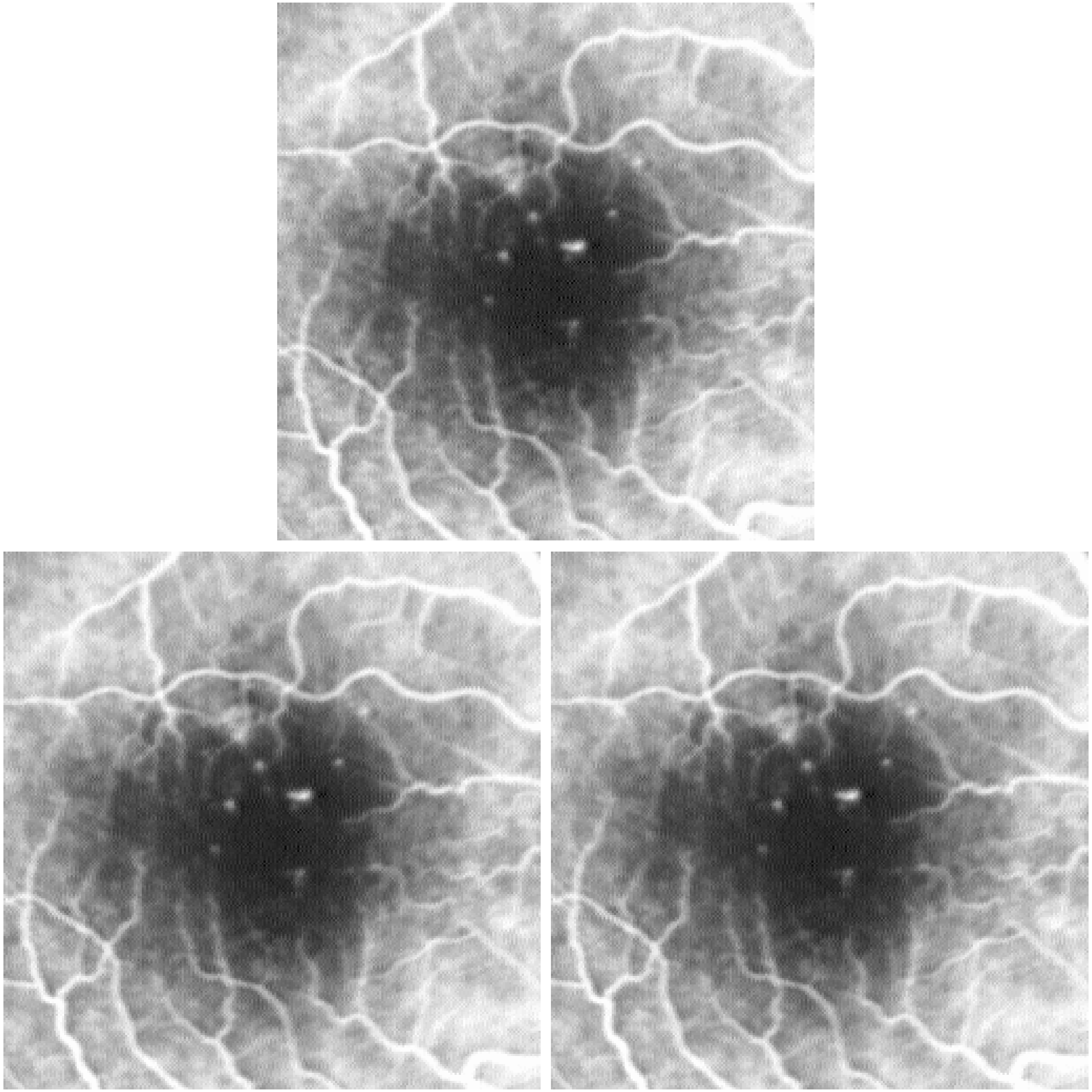
ASF with an hexagon (maximal size = 5)

Application to alternate sequential filters



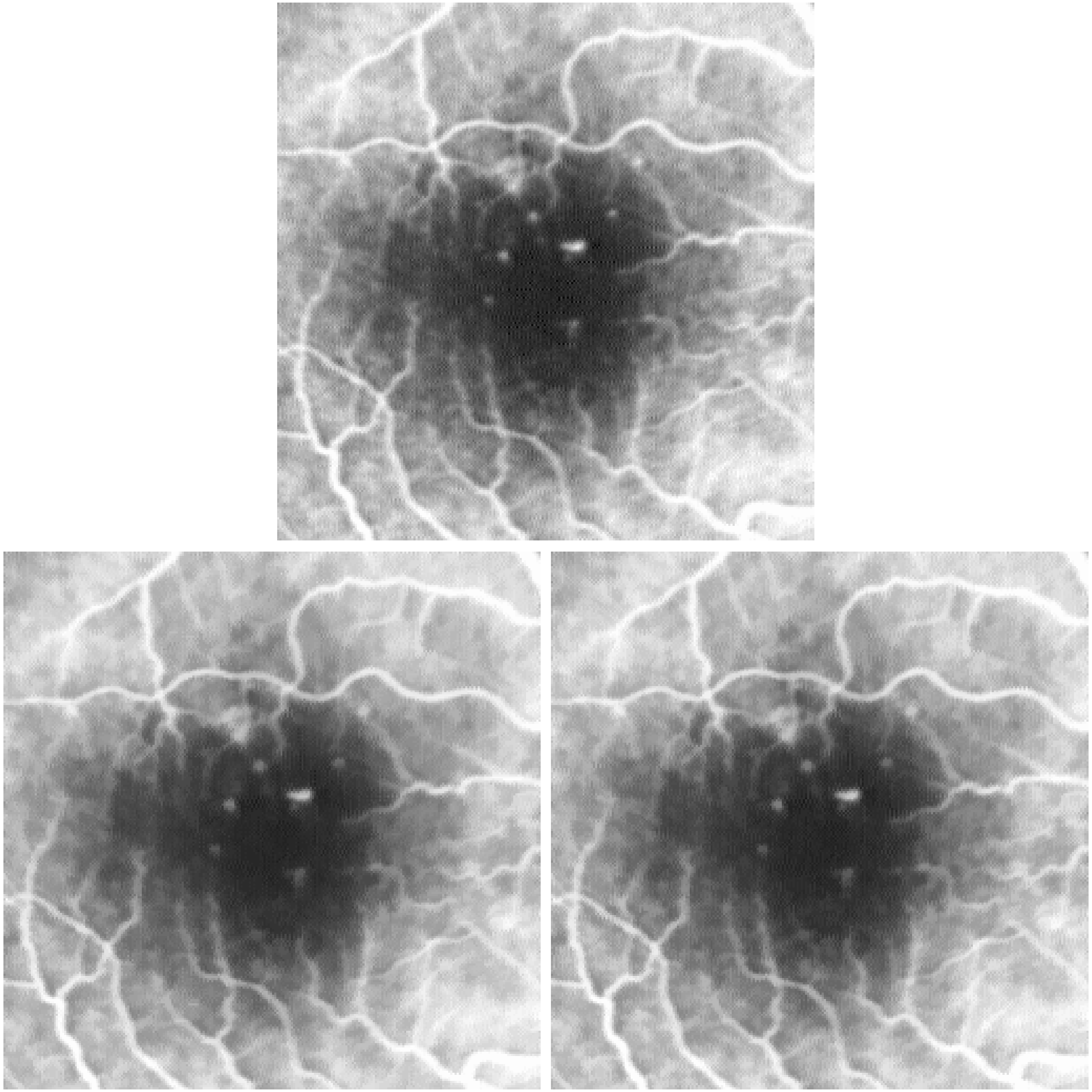
ASF with an hexagon (maximal size = 9)

Application to alternate sequential filters



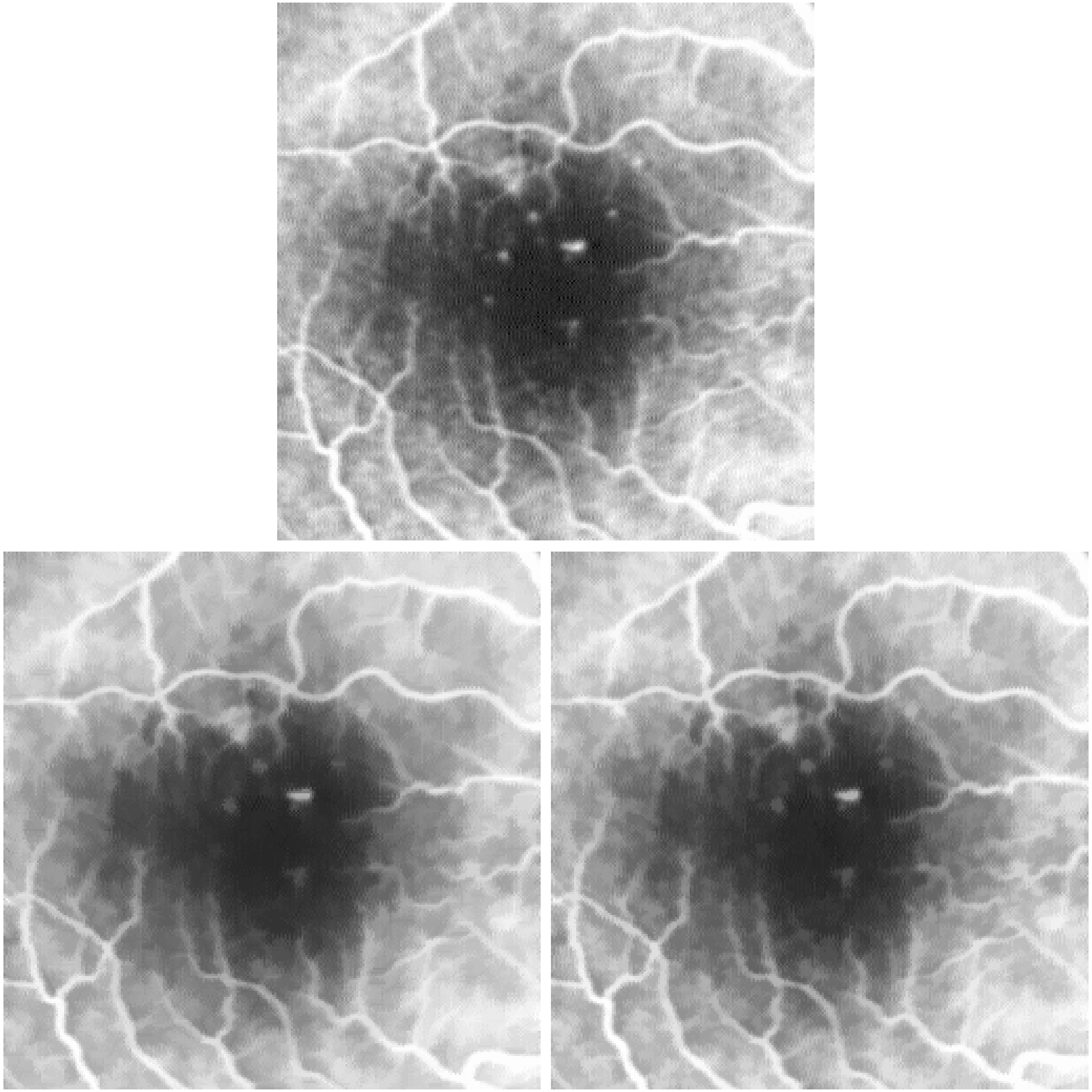
ASF with segments (maximal size = 1)

Application to alternate sequential filters



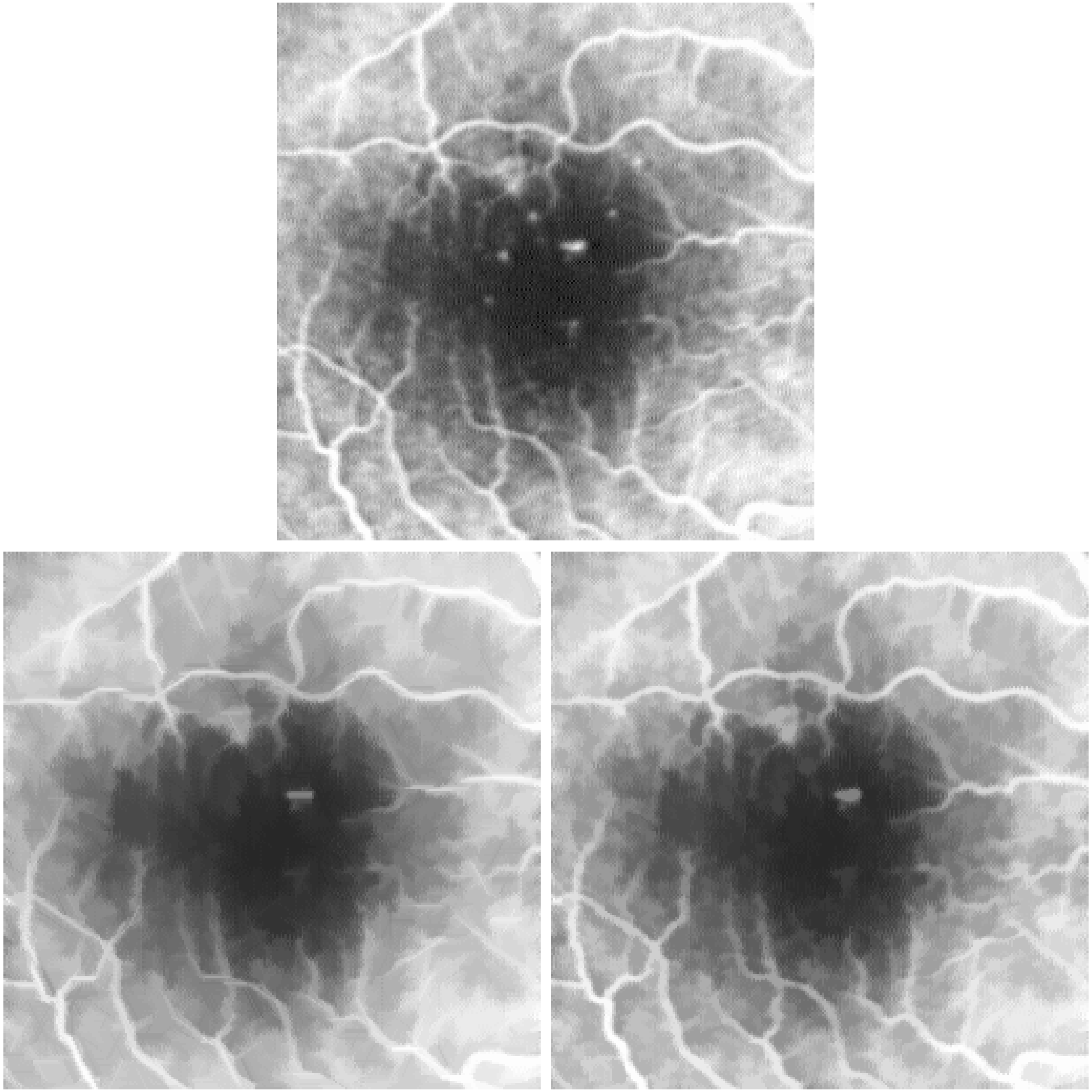
ASF with segments (maximal size = 3)

Application to alternate sequential filters



ASF with segments (maximal size = 5)

Application to alternate sequential filters



ASF with segments (maximal size = 9)

Regional maxima

X regional maximum of f if

$$\forall x \in X, f(x) = \lambda \text{ et } X = CC(f_\lambda)$$

Computation of regional maxima:

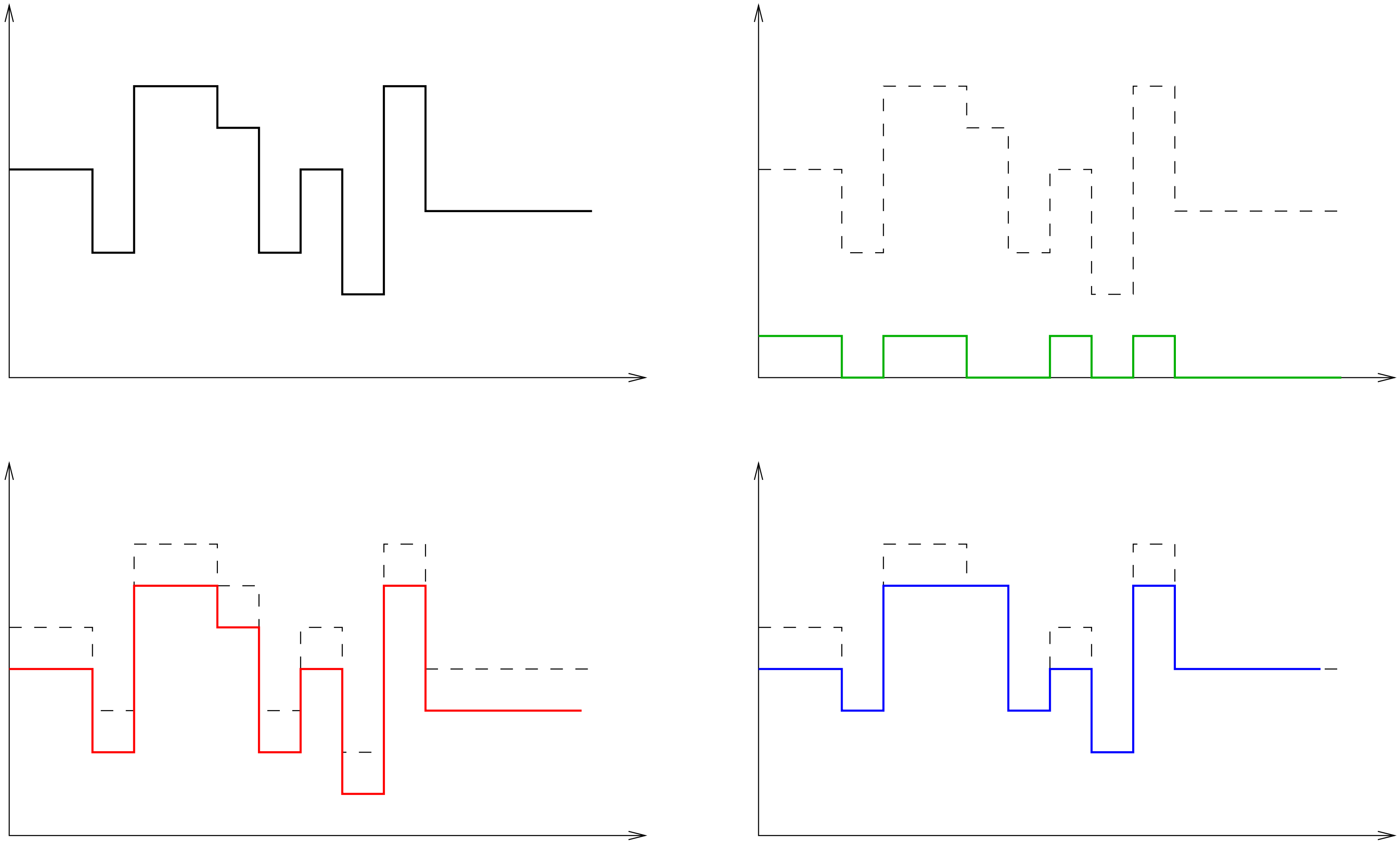
$$f - D_f^\infty(f - 1)$$

h-maxima (grey level dynamics):

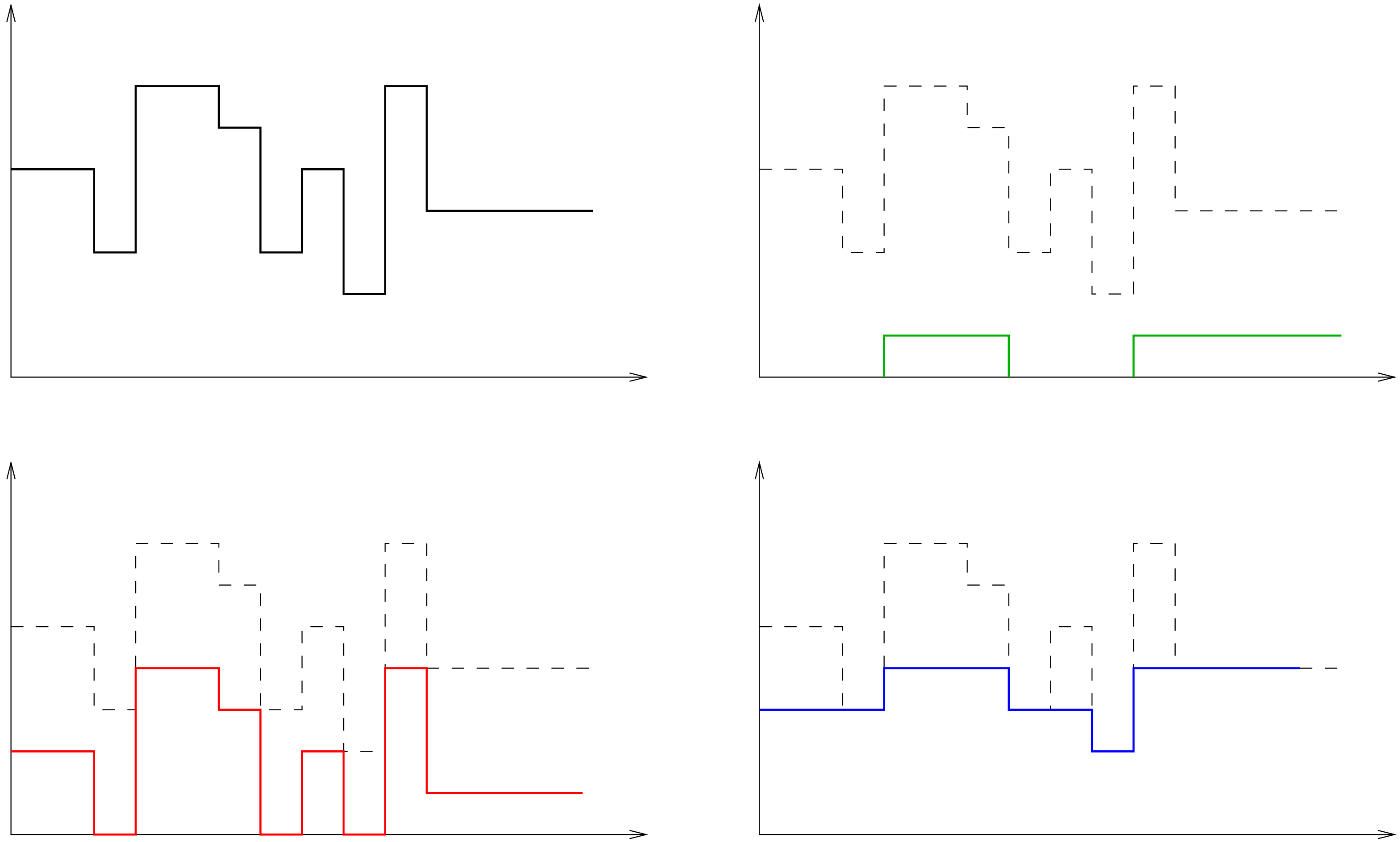
$$f - D_f^\infty(f - h)$$

$\Rightarrow$ robust maxima

Regional maxima: example



Robust maxima: example



Skeleton by influence zones

$X = \bigcup_i X_i$

Influence zone of X_i in X^C :

$$ZI(X_i) = \{x \in X^C \ / \ d(x, X_i) < d(x, X \setminus X_i)\}$$

Skeleton by influence zones:

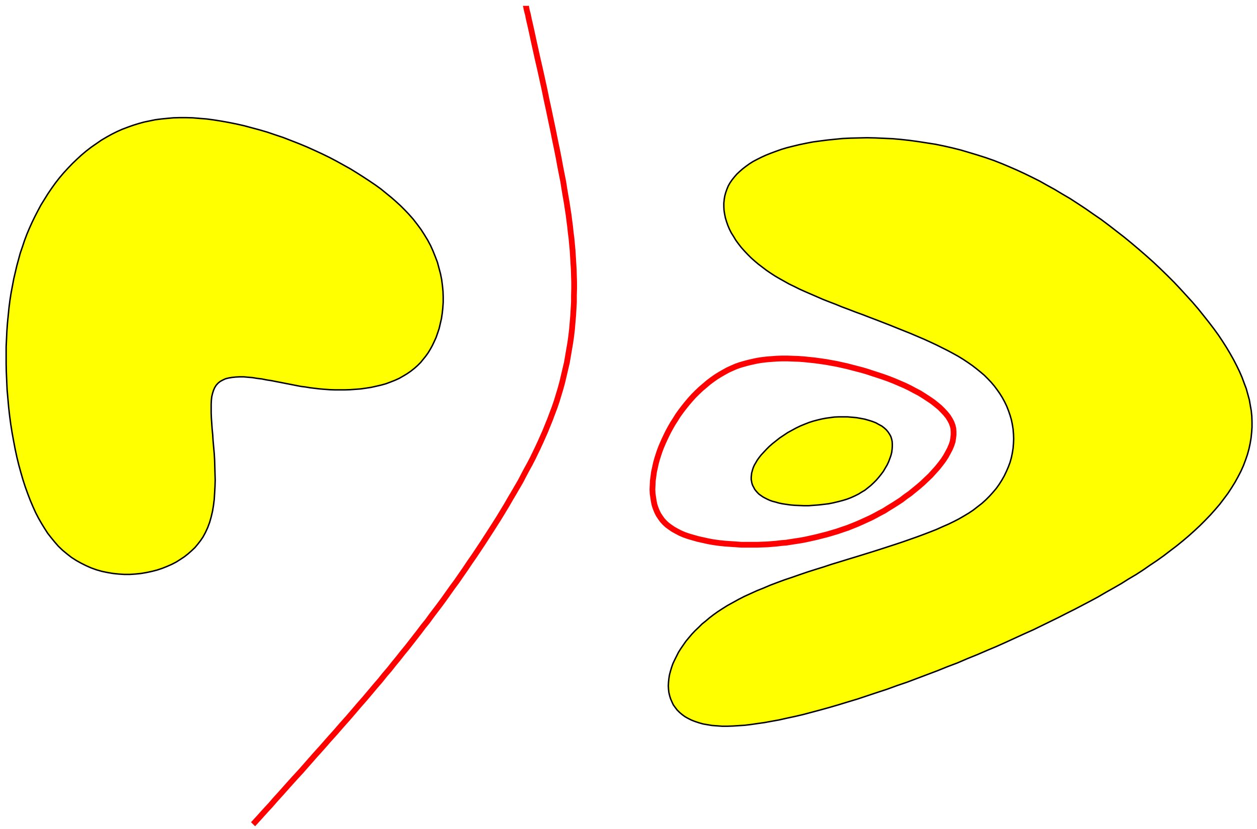
$$\text{Skiz}(X) = (\bigcup_i ZI(X_i))^C$$

= generalized Voronoï diagram

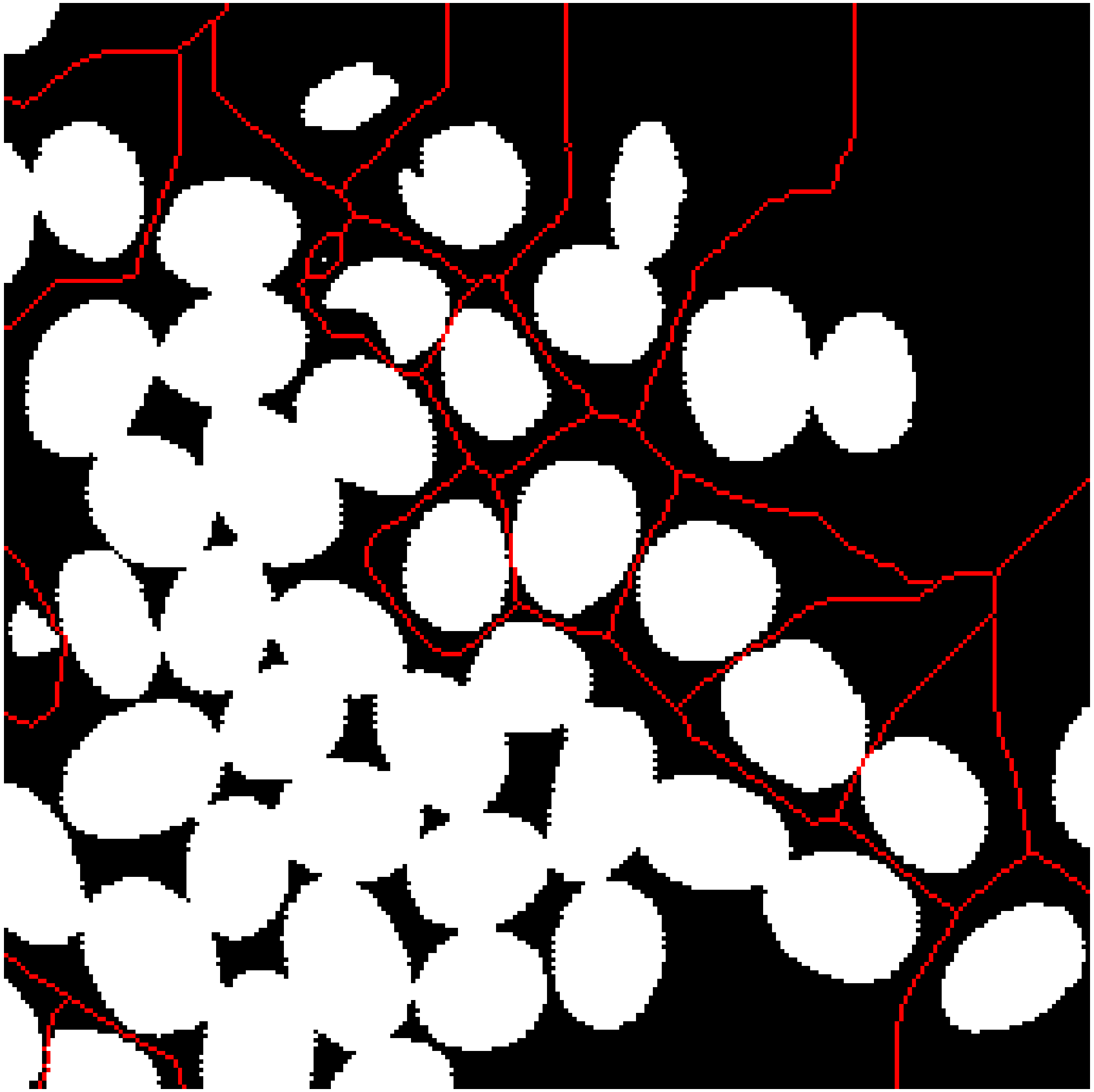
Properties:

- $\text{Skiz}(X) \subseteq Skel(X^C)$
- Skiz is not necessarily connected (even if X^C is)

Skeleton by influence zones: examples

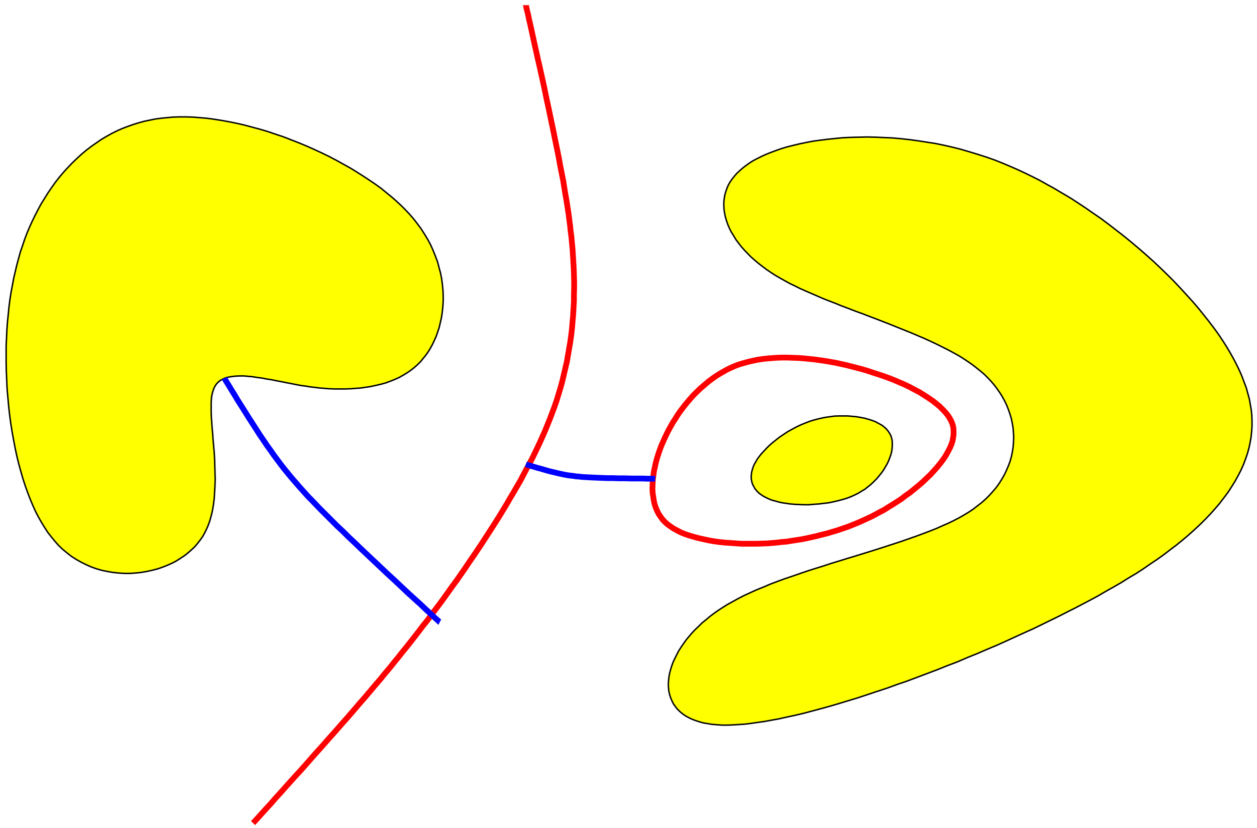


Skeleton by influence zones: examples



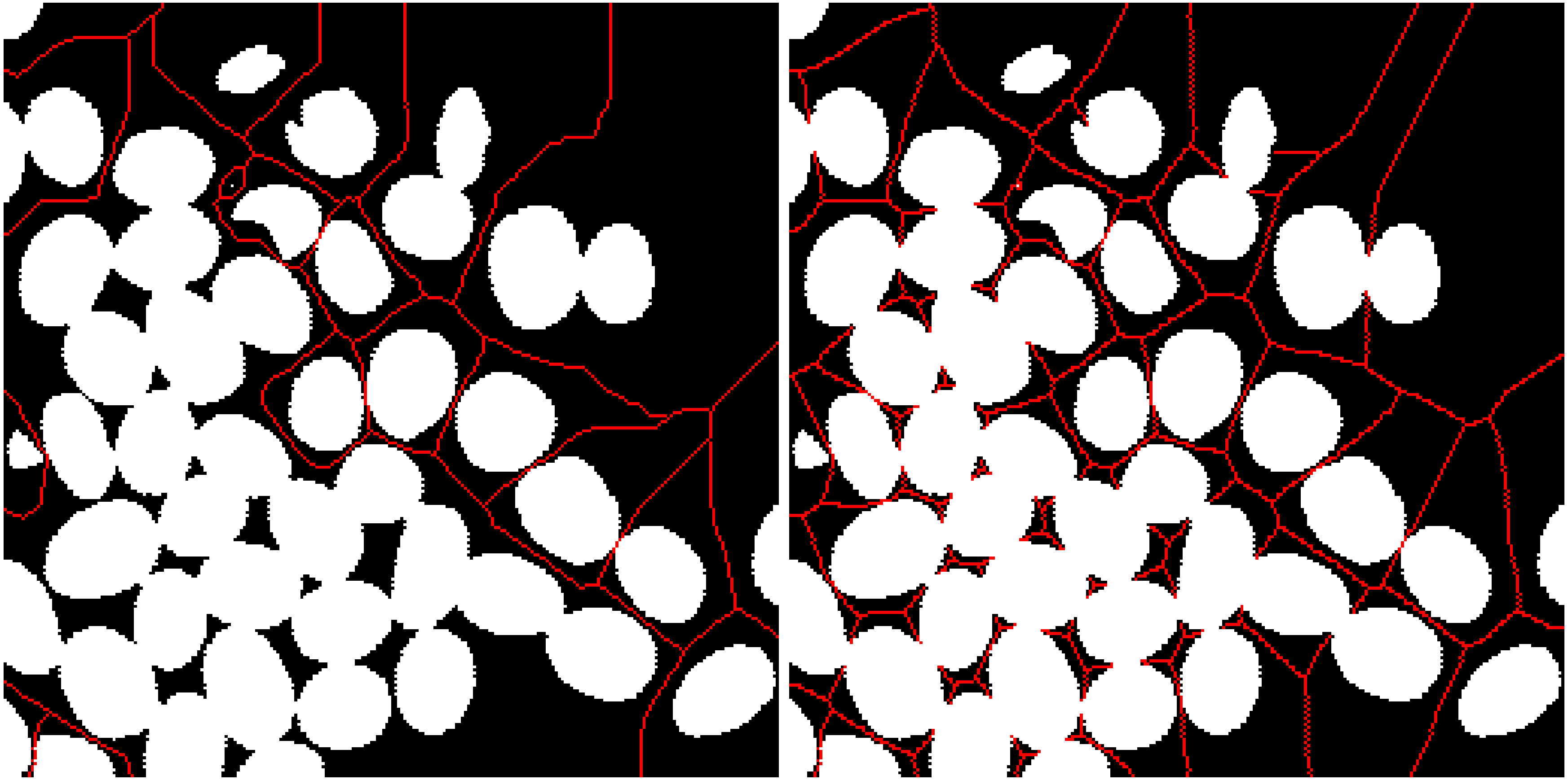
Skeleton and skeleton by influence zones

$$\mathsf{Skiz}(X) \subseteq r(X^C)$$



Skeleton and skeleton by influence zones

$$\mathsf{Skiz}(X) \subseteq r(X^C)$$



Geodesic skeleton by influence zones

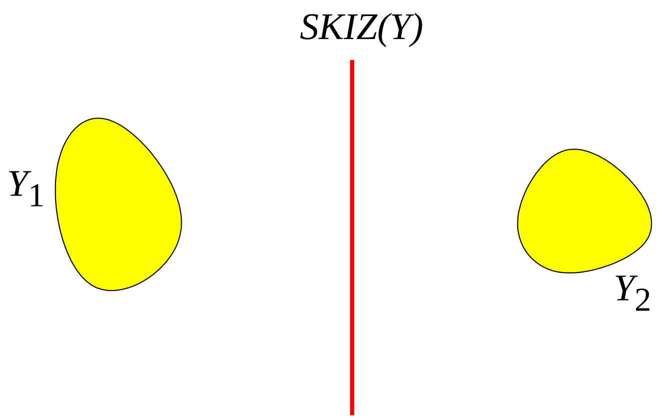
$Y = \cup_i Y_i$

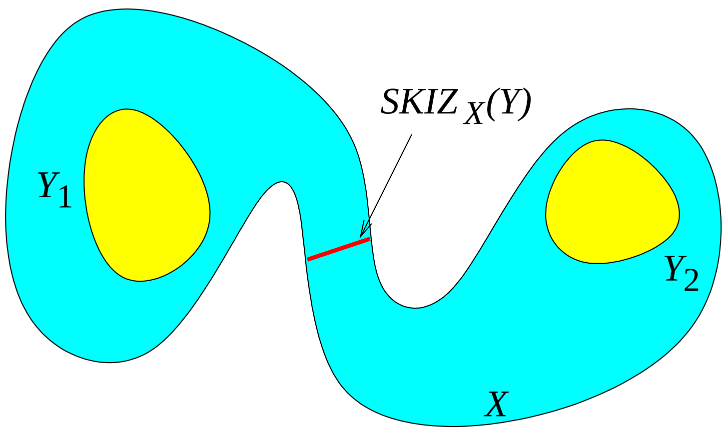
Geodesic influence zone of Y_i conditionally to X :

$$ZI_X(Y_i) = \{x \in X, d_X(x, Y_i) < d_X(x, Y \setminus Y_i)\}$$

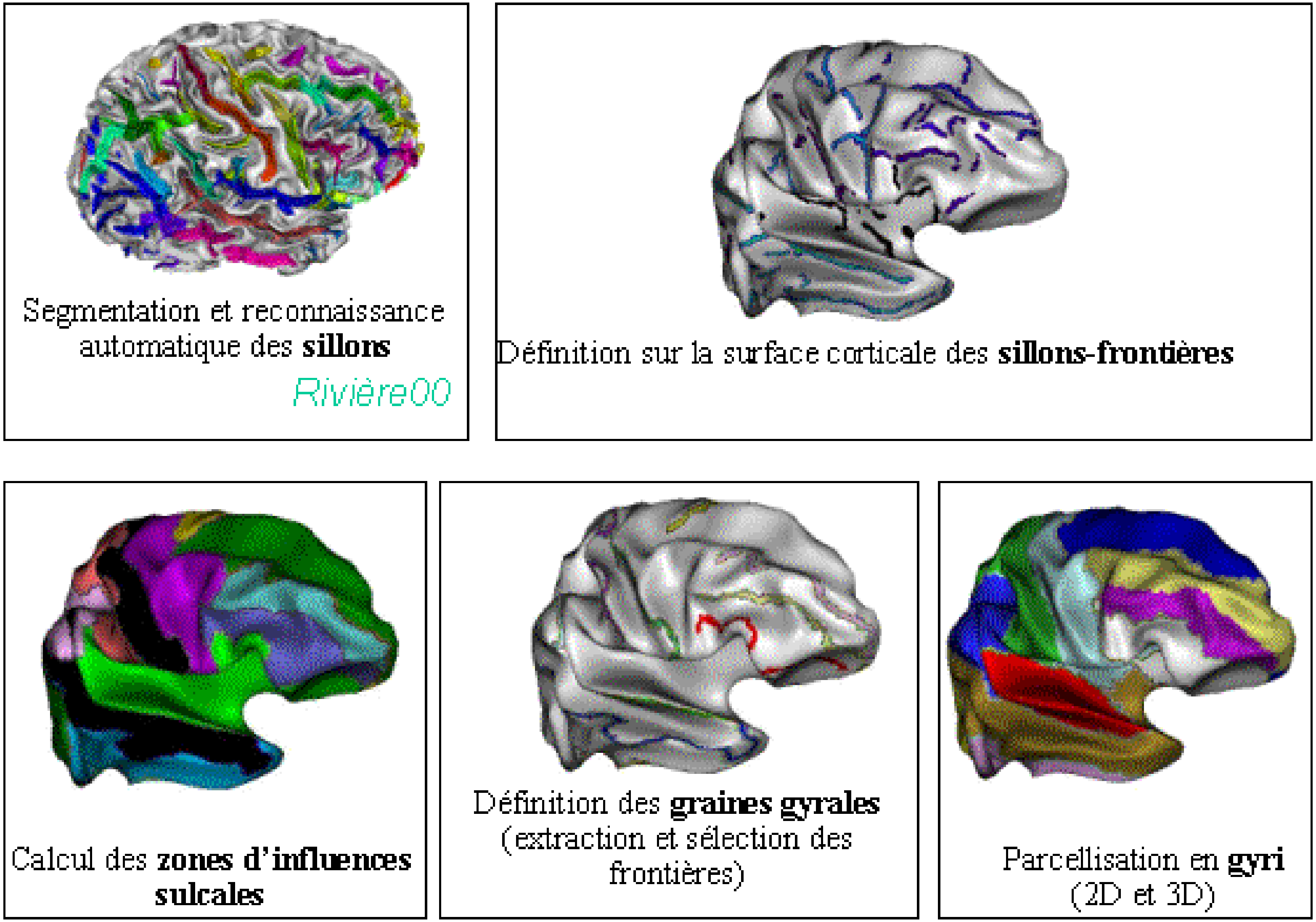
Geodesic skeleton by influence zones:

$$SKIZ_X(Y) = X \setminus \bigcup_i ZI_X(Y_i)$$

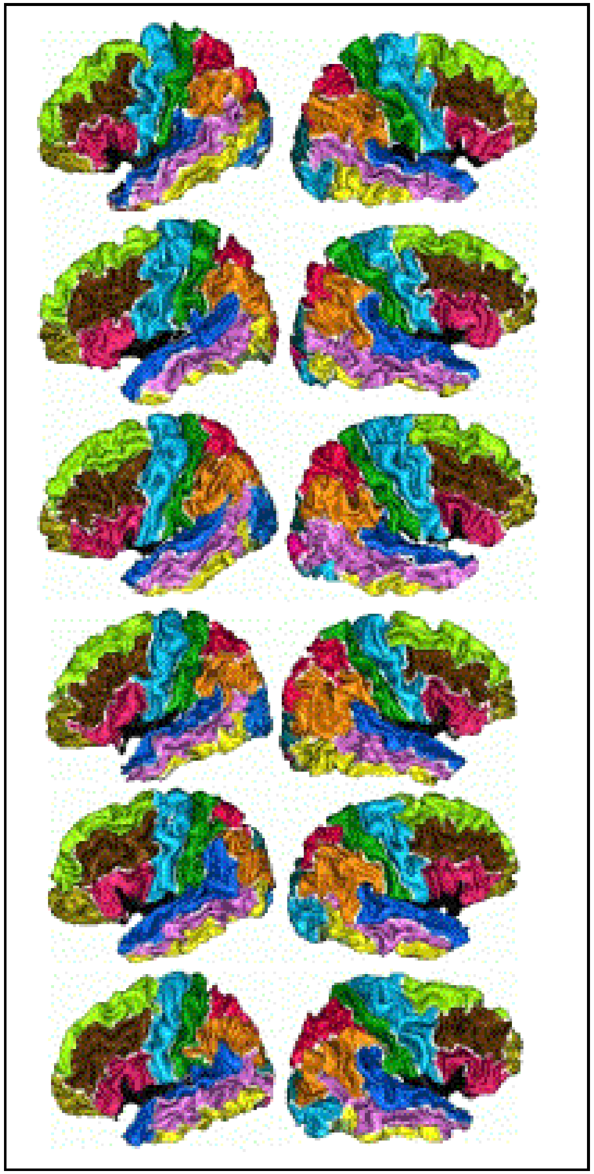


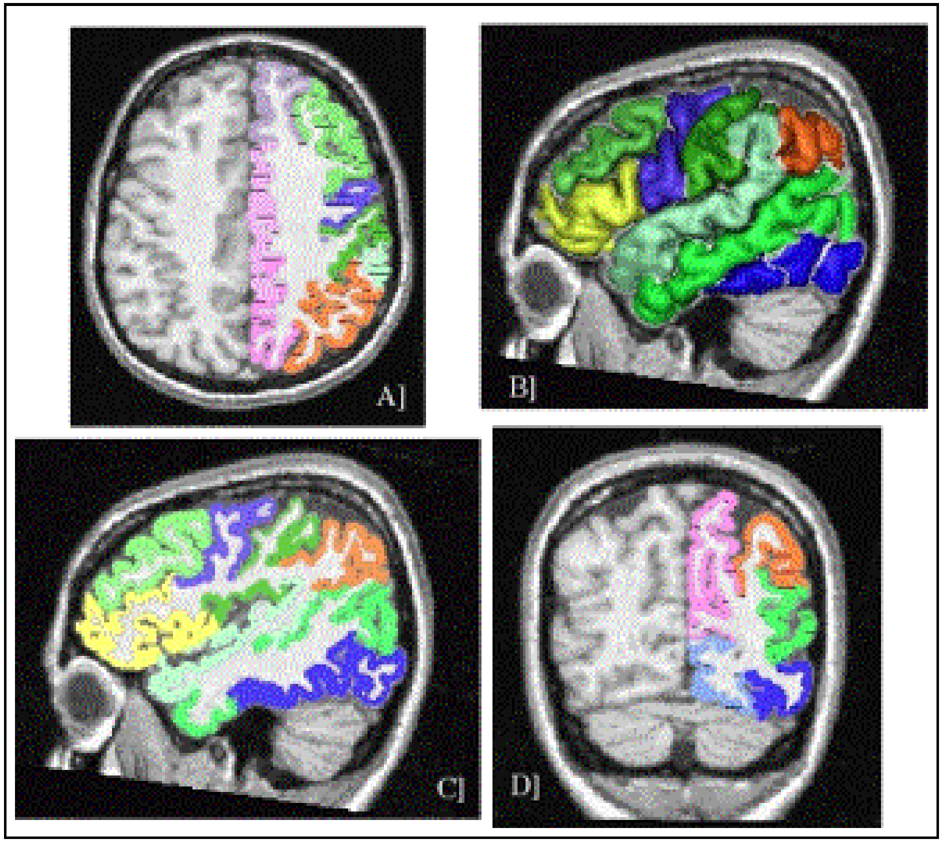


Cortex segmentation
(PhD of Arnaud Cachia)



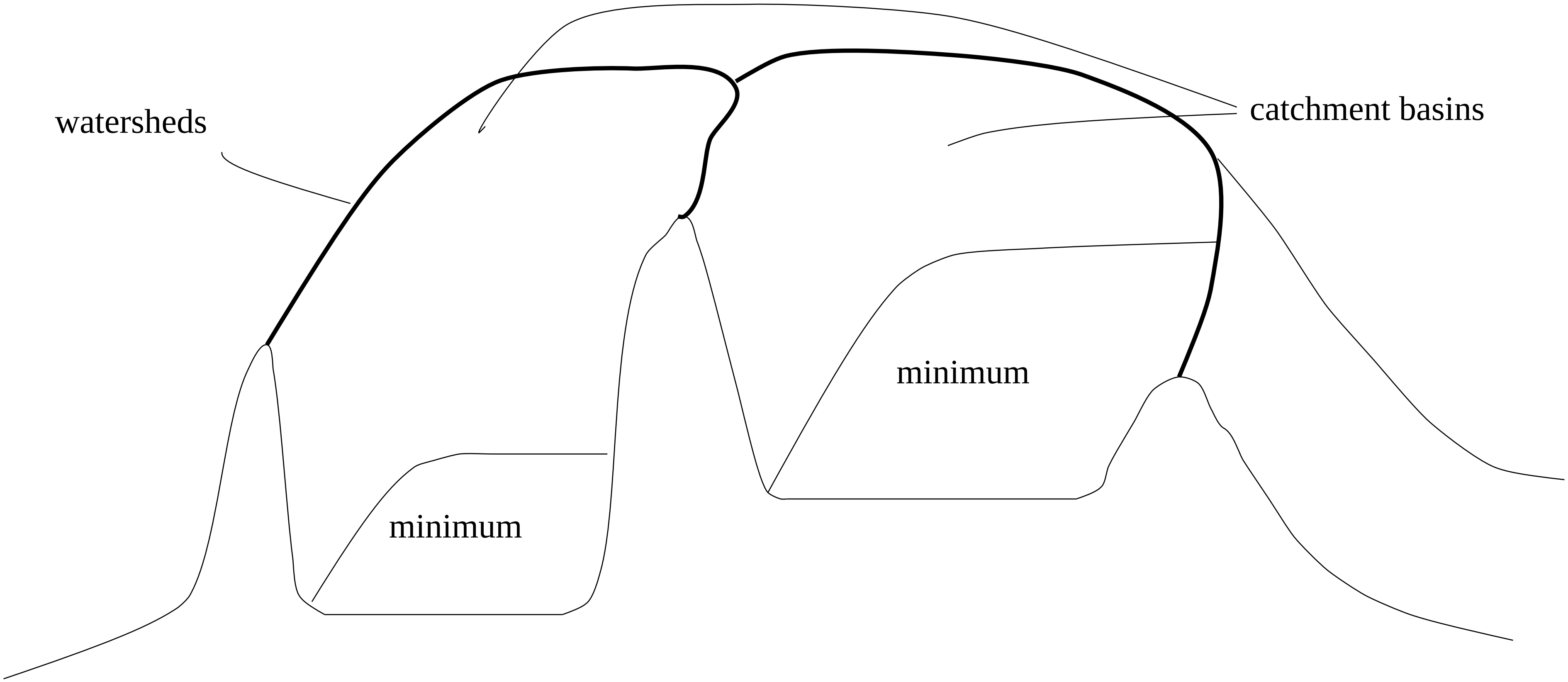
Cortex segmentation
(PhD of Arnaud Cachia)





Parcellisation volumique
(diagramme de Voronoï calculé
dans le ruban cortical 3D)

Watersheds



Watersheds: definition

Steepest descent:

$$Desc(x) = \max\{\frac{f(x) - f(y)}{d(x,y)},\ y \in V(x)\}$$

Ramp of a path $\pi = (x_0, ...x_n)$:

$$T_f(\pi) = \sum_{i=1}^{n} d(x_{i-1}, x_i) Cost(x_{i-1}, x_i)$$

with

$$Cost(x,y) = \begin{cases} Desc(x) & \text{if } f(x) > f(y) \\ Desc(y) & \text{if } f(y) > f(x) \\ (Desc(x) + Desc(y))/2 & \text{if } f(y) = f(x) \end{cases}$$

Watersheds: definition

Topographic distance

$$T_f(x,y) = \inf\{T_f(\pi), \pi = (x_0 = x, x_1, ..., x_n = y)\}$$

(equals 0 on a plateau)

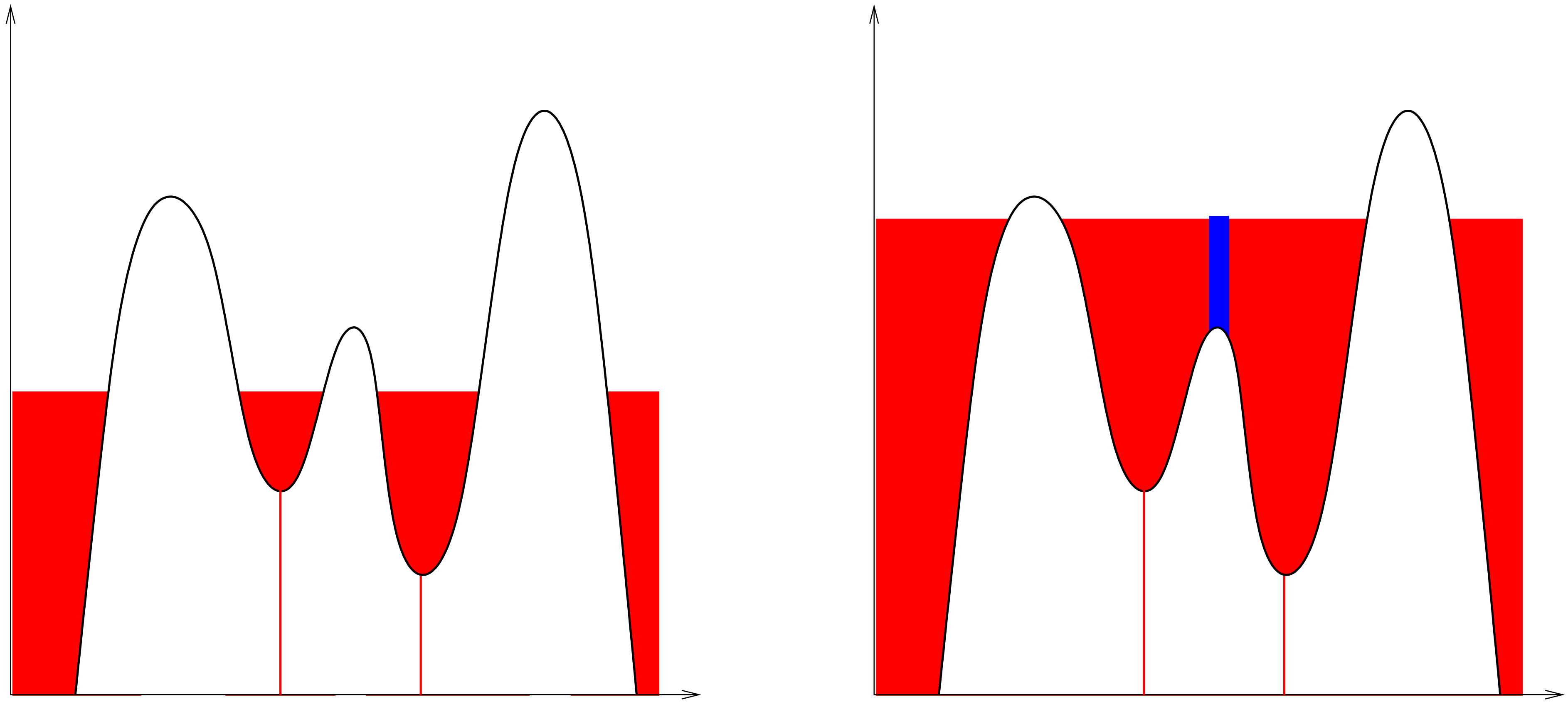
Catchment basin associated to the regional minimum M_i:

$$BV(M_i) = \{x,\ \forall j \neq i, T_f(x, M_i) < T_f(x, M_j)\}$$

Watersheds:

$$LPE(f) = [\cup_i BV(M_i)]^C$$

Approach by immersion



Construction of the watersheds

f such that $f(x) \in [h_{\min}, h_{\max}]$, $f^h = \{x,\ f(x) \leq h\}$

$$
\begin{aligned}
X_{h_{\min}} &= f^{h_{\min}} \\
X_{h+1} &= MinReg_{h+1}(f) \cup ZI_{f^{h+1}}(X_h) \\
BV &= X_{h_{\max}} \\
LPE(f) &= X^C_{h_{\max}}
\end{aligned}
$$

Illustration of the algorithm

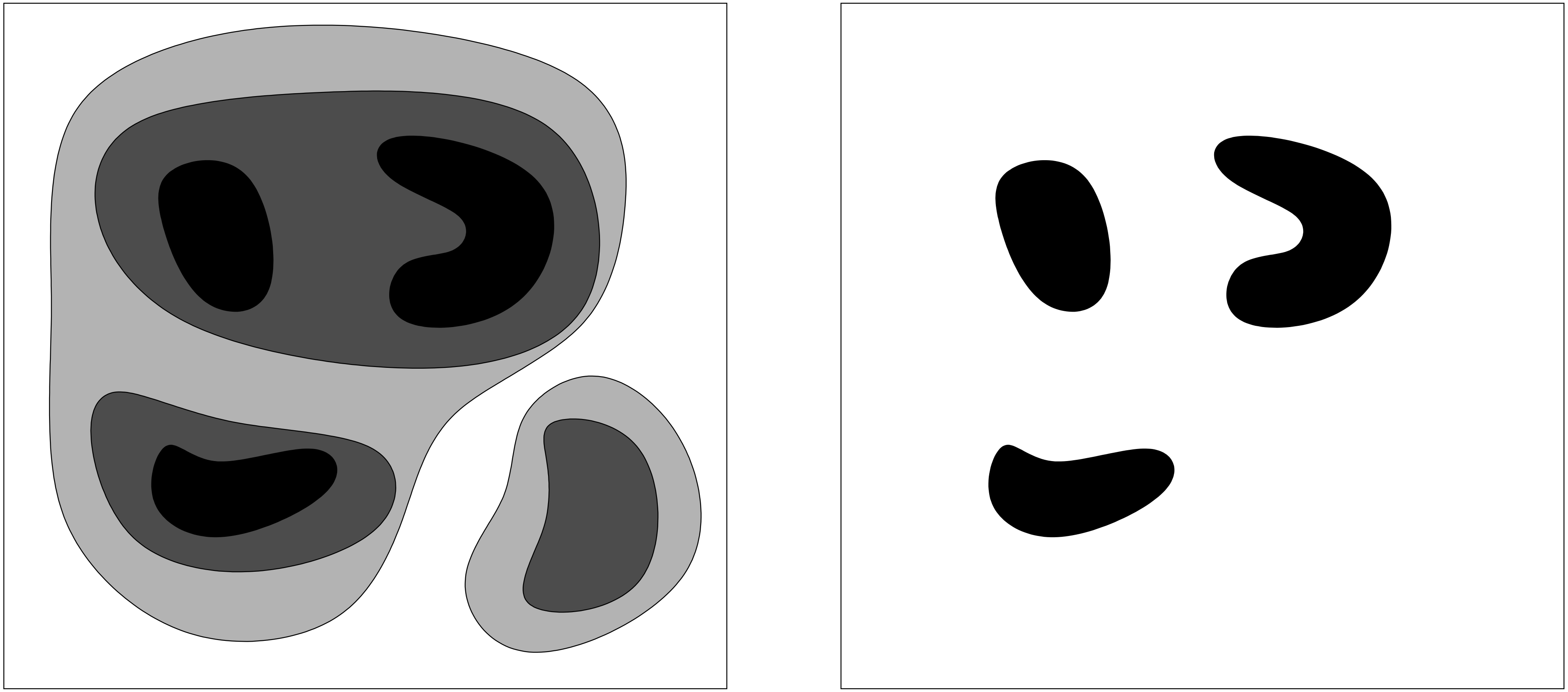


Illustration of the algorithm

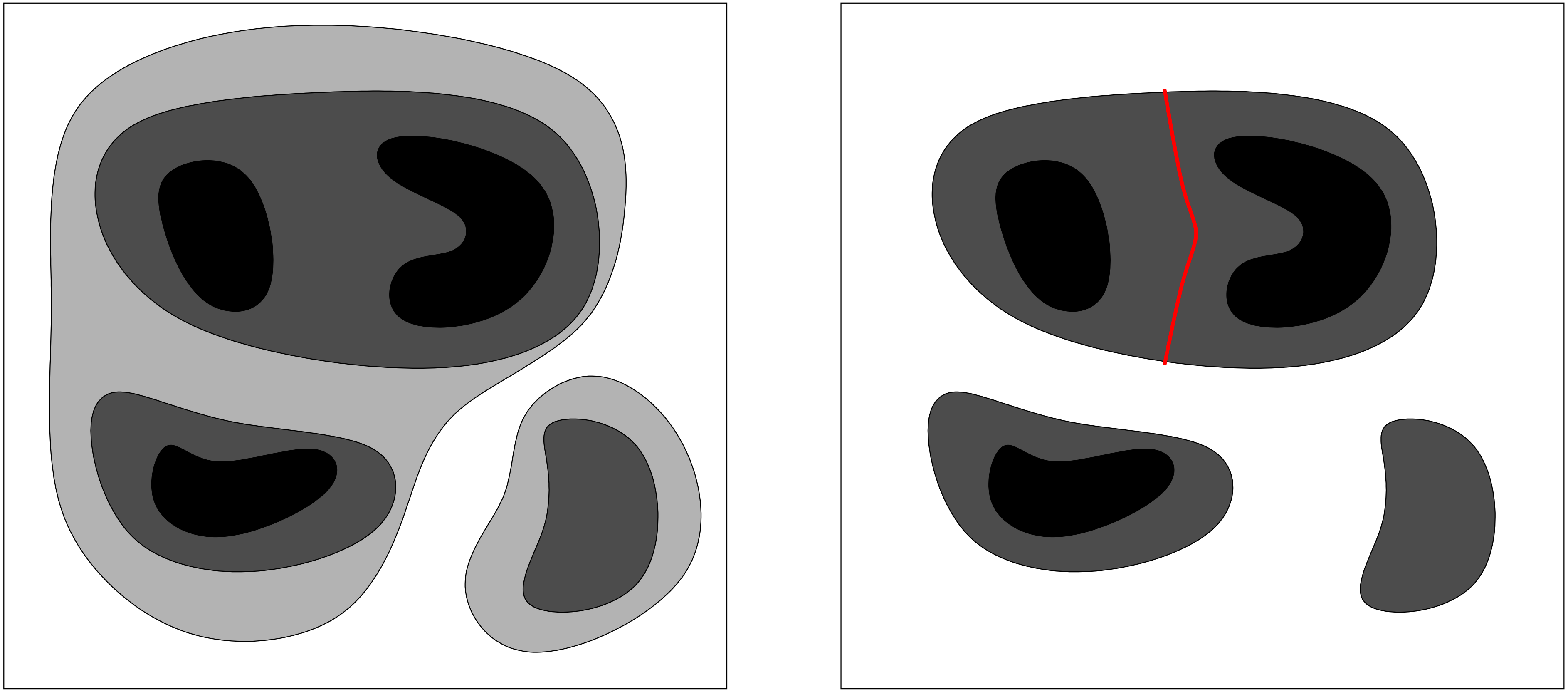


Illustration of the algorithm

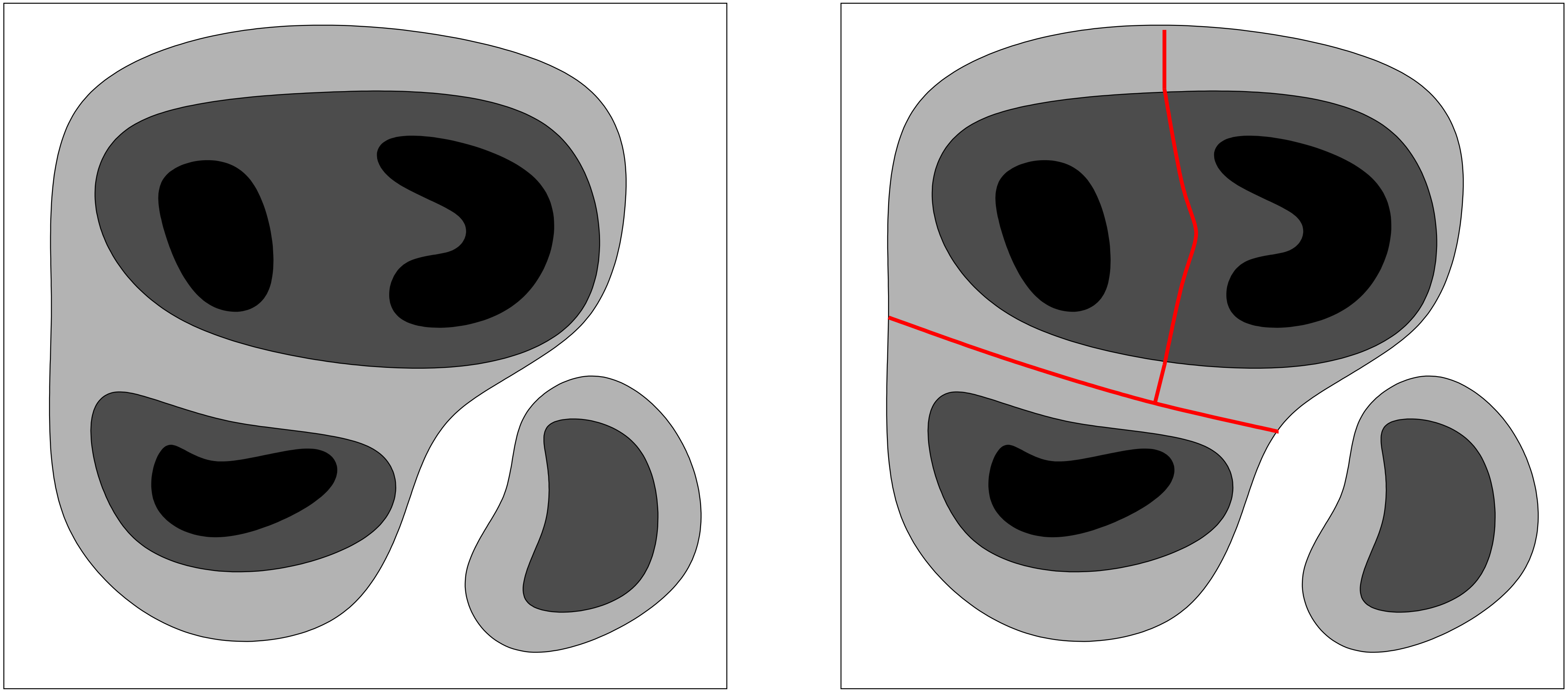
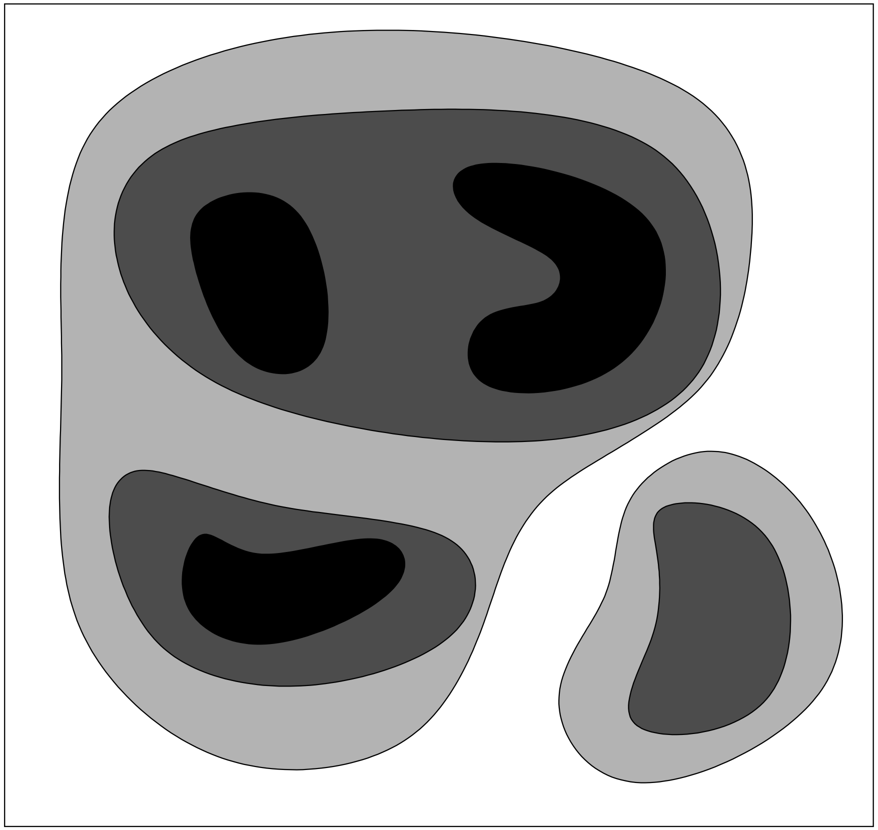
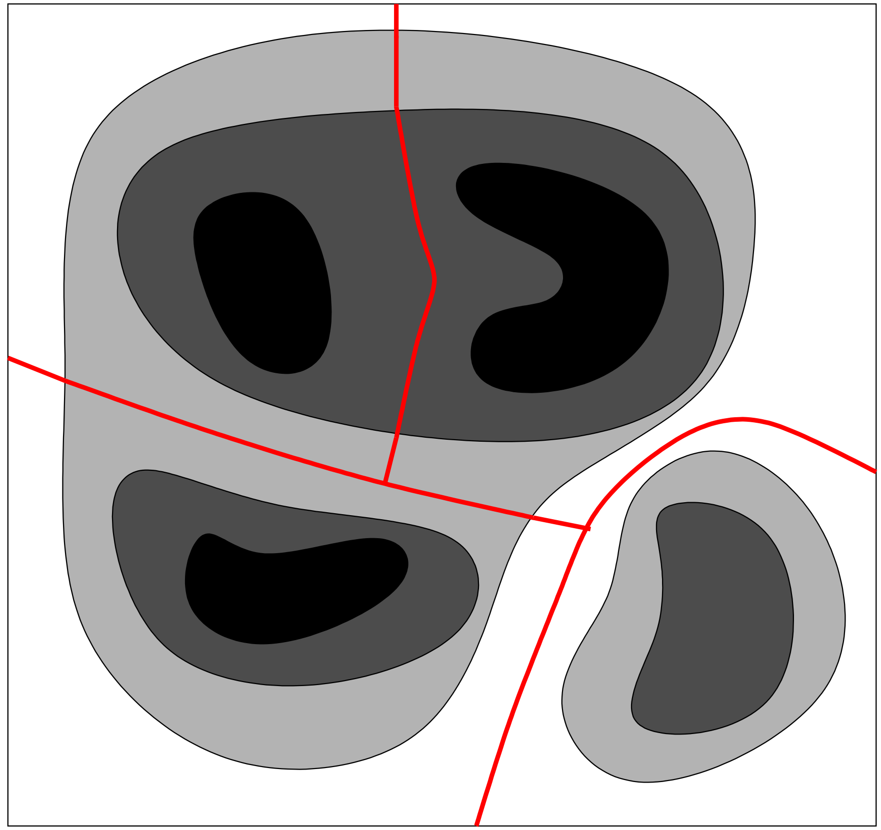
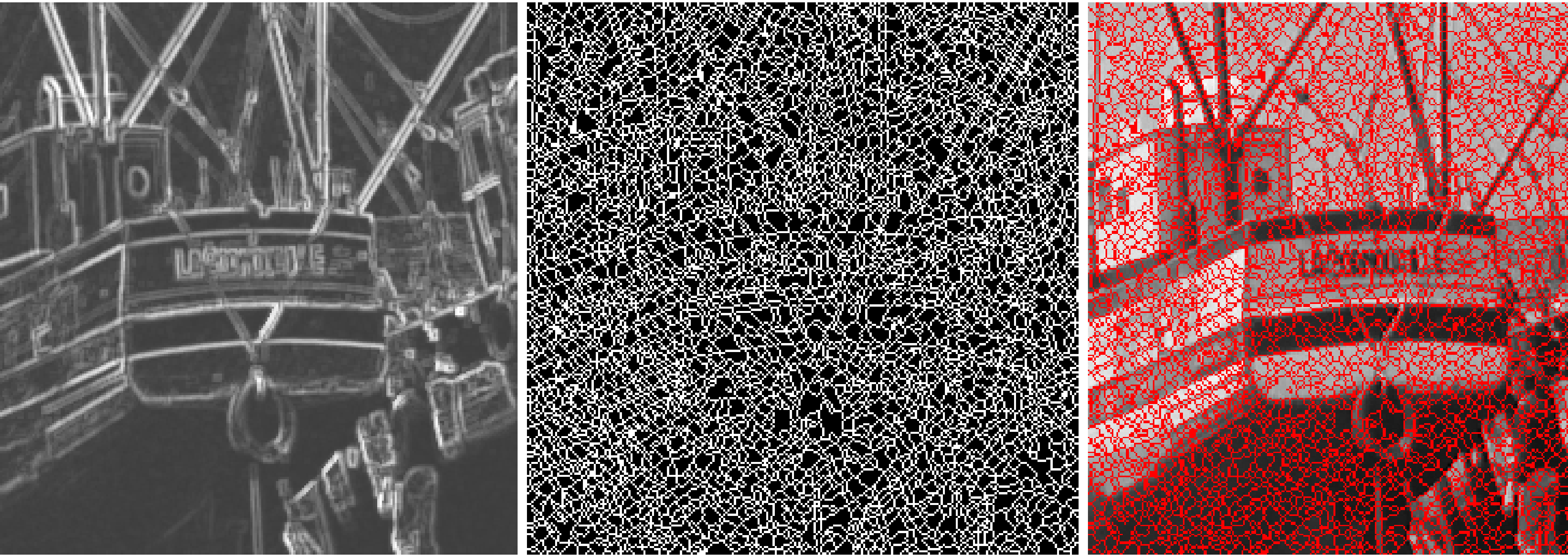


Illustration of the algorithm

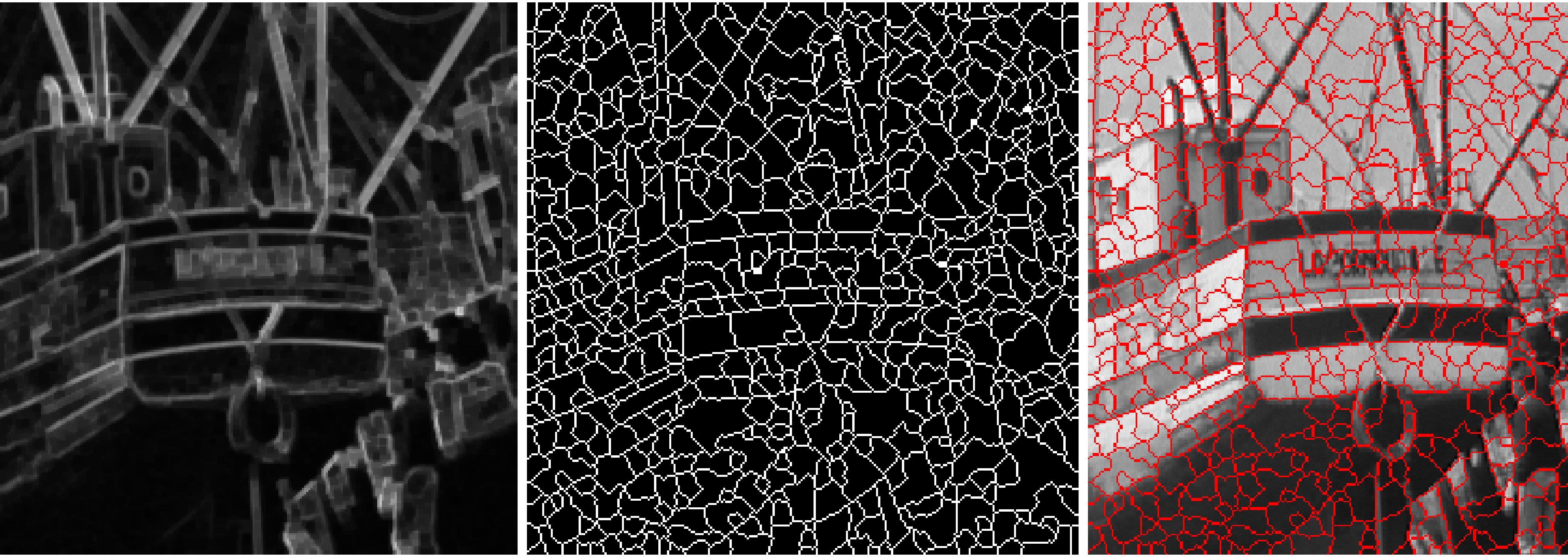




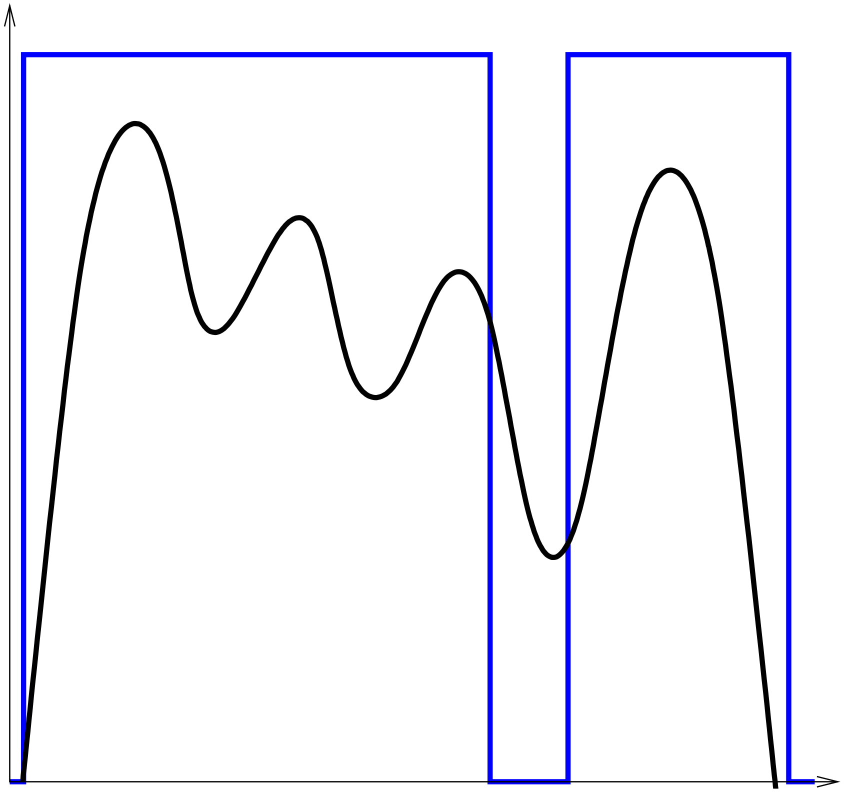
Watersheds and oversegmentation

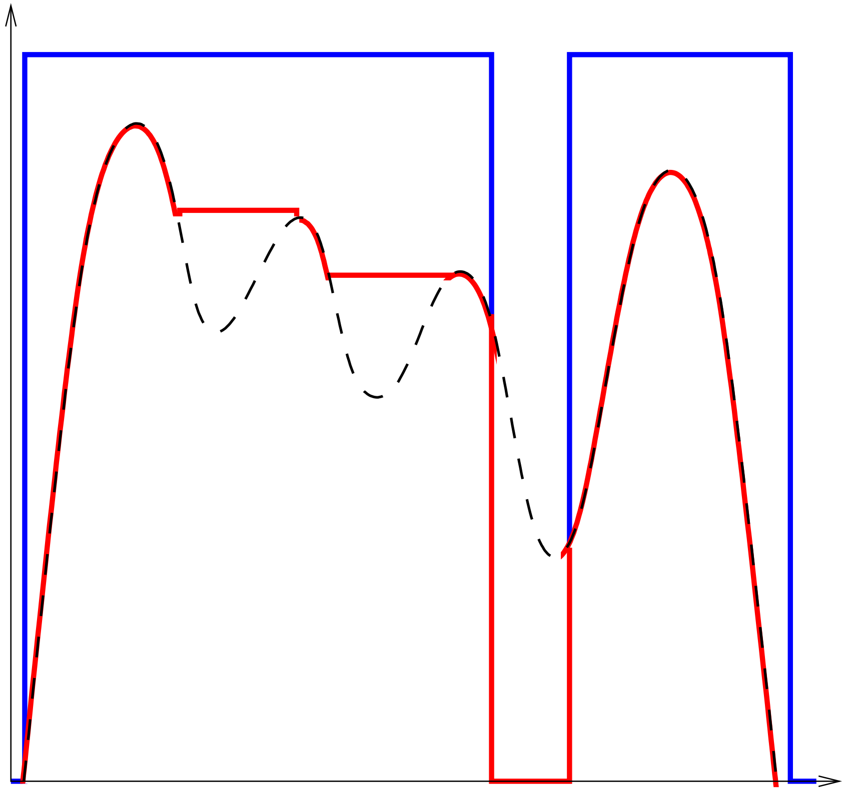


Watersheds and oversegmentation



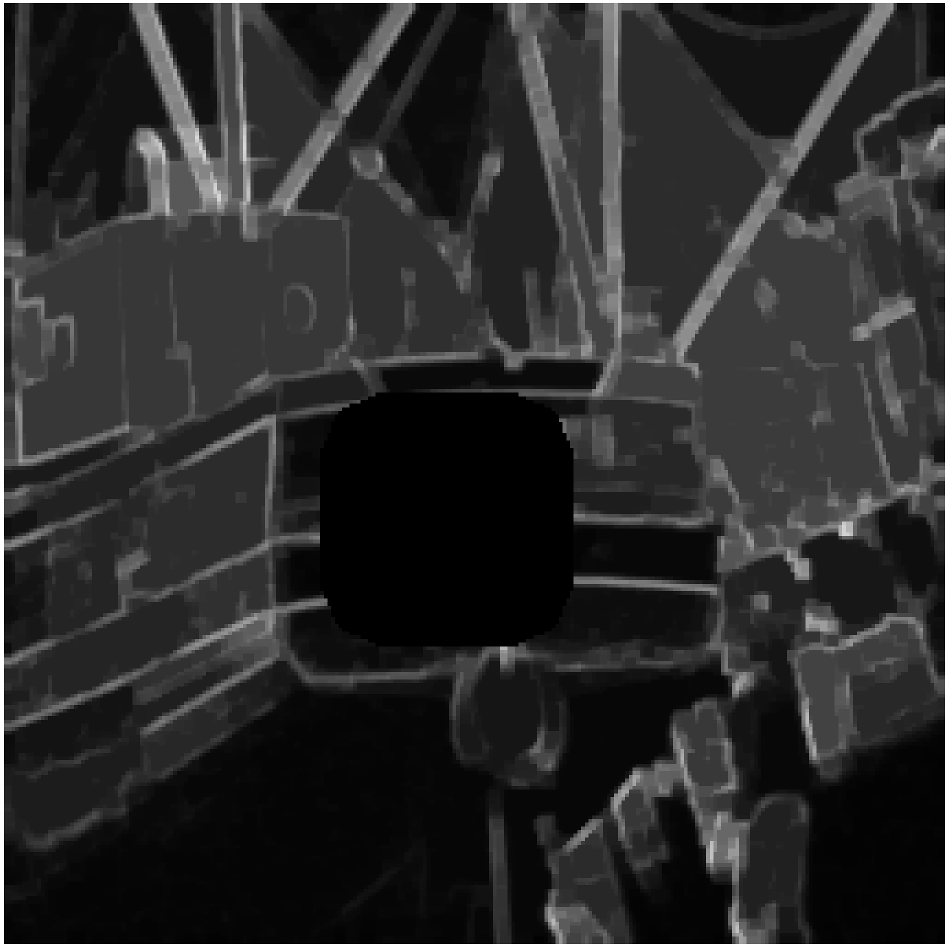
Geodesic erosion in order to impose markers



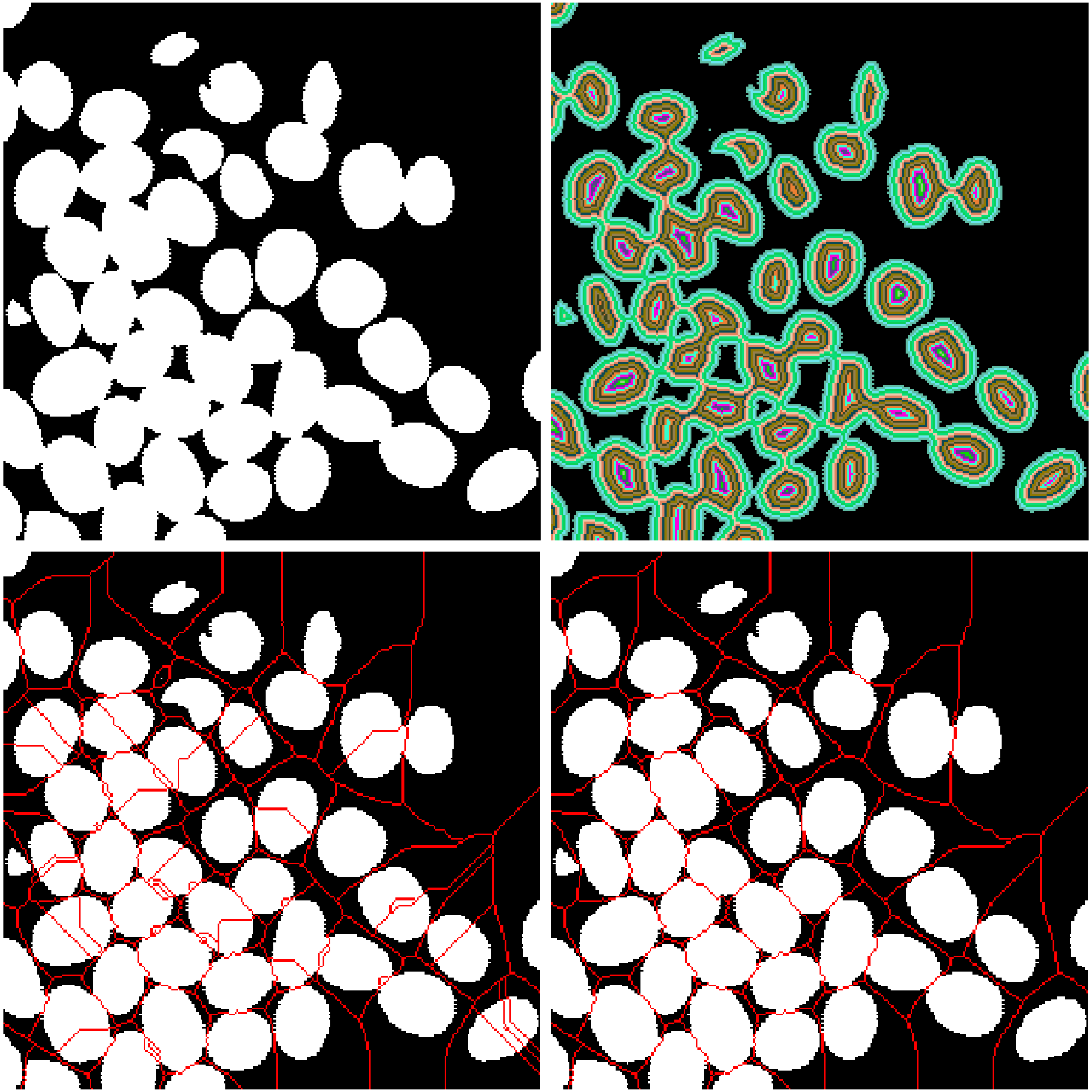


Watersheds constraint by markers

f: function on which watersheds should be applied
g: marker function (selects regional minima)
Reconstruction: $E_{f \wedge g}(g, B_\infty)$ (only the selected minima)



Separation of connected binary objects



And much more...